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INVERSE PROBLEM FOR THE VIBRATING BEAM IN THE FREE–CLAMPED CONFIGURATION

By V. BARCILON
Department of the Geophysical Sciences, The University of Chicago, Chicago, Illinois 60637, *U.S.A.*

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We consider the problem of reconstructing the flexural rigidity $r(x)$ and the density $\rho(x)$ of a beam. The unknown beam is assumed to have a free left end and a clamped right end. The data consist of the displacement and angle of the centre line of the free left end after an initial impulse. The information content of this seismogram-like impulse response is equivalent to three spectra $\{\omega_n\}$, $\{\nu_n\}$, $\{\mu_n\}$ and two gross constants F_1, F_2. These data do not specify the structure of the vibrating beam uniquely, but rather a class of beams. All the beams in this class share the same structure over that portion of their length which is actually set in motion; they can differ over the portion that is stationary. A method for constructing $r(x)$ and $\rho(x)$ is presented. It consists of two steps. First $\rho(x)$ and $r(x)$ are determined over a small interval $(0, x)$ adjacent to the free left end. Next, this known portion of the beam is stripped off and the response of the resulting truncated beam is computed via the initial data. The procedure is then repeated. Finally, the question of the existence of a solution is discussed. More specifically, conditions on $\{\omega_n\}$, $\{\nu_n\}$ and $\{\mu_n\}$ are given that ensure that $r(x)$ and $\rho(x)$ are physically meaningful.

Introduction

This paper is devoted to a review of the current status of the inverse problem for a vibrating beam. This inverse problem consists of reconstructing the flexural rigidity $r(x)$ and the density $\rho(x)$ as functions of position x, from data associated with natural frequencies of vibration of the beam.

The reconstruction of $r(x)$ and $\rho(x)$ is made without assuming any *a priori* knowledge about the structure of the beam. Consequently, this approach differs from the traditional approach for the geophysical problem dealing with the reconstruction of the internal structure of the Earth from data on toroidal and spheroidal oscillations. There, the spectral data are used to *correct* a model that incorporates knowledge from travel time, surface waves, etc. Thus, the Backus–Gilbert technique, which is ideally suited for such correcting tasks, is not suitable here.

This geophysical problem is a primary motivation for the investigation of the inverse problem for a vibrating beam. Indeed, if one neglects the gravitational force, the rotation, the oblateness, etc. and considers the Earth as an elastic, radially stratified sphere, then its normal modes are of the two types previously alluded to, namely the spheroidal modes and the toroidal modes. The latter are governed by a second-order equation of Sturm–Liouville type whereas the former are governed by a fourth-order system. The bulk and shear modulii as well as the density, which characterize the elastic properties of this idealized Earth, are assumed unknown. The inverse problem consists of retrieving these characteristics. To understand this difficult inverse problem, it might be helpful to consider simpler inverse problems, and, in particular, simpler inverse problems associated with fourth-order equations since the inverse Sturm–Liouville problem is well understood. The inverse problem for the vibrating beam *as described by the Bernoulli–Euler theory* is the ideal candidate. The shortcomings of this theory ought not to preoccupy us: indeed, we are primarily interested in dealing with the simplest inverse fourth-order eigenvalue problem, and only secondarily in the beam *per se*.

This problem can also be viewed as an example of an inverse problem associated with a class of differential operators of order $2n$, namely

$$\mathscr{L}_{2n} = (-1)^n \alpha_0 \frac{d}{dx} \alpha_1 \frac{d}{dx} \cdots \frac{d}{dx} \alpha_n \frac{d}{dx} \cdots \frac{d}{dx} \alpha_1 \frac{d}{dx} \alpha_0,$$

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where all the α are positive. For certain boundary conditions, the Green functions associated with these operators are oscillating kernels. The theory of such kernels plays an important role in the solution of the inverse problem.

The outline of the paper is as follows. In §1 we shall introduce various fundamental solutions of the equation for the vibrating beam, and state some of their properties. We shall also introduce five eigenvalue problems associated with the same beam but with different boundary conditions at the left end. A central role will be played by the problem for the beam in the free–clamped configuration, namely

$$\frac{d^2}{dx^2}\left[r(x)\,\frac{d^2\psi_n}{dx^2}\right] = \omega_n^2\rho(x)\,\psi_n, \quad 0 < x < L,$$

$$\psi'' = (r\psi'')' = 0 \quad \text{at} \quad x = 0,$$

$$\psi = \psi' = 0 \quad \text{at} \quad x = L.$$

The question of the uniqueness of the inverse problem is taken up in §2. More specifically, we shall investigate whether the displacement and slope of the central line at the left for all time, i.e. the impulse response, determine $r(x)$ and $\rho(x)$ uniquely. We shall see that the information contained in this impulse response is equivalent to the knowledge of the spectrum $\{\omega_n\}$ of the free–clamped beam and of two other spectra, $\{\nu_n\}$ and $\{\mu_n\}$ defined in §1, as well as of two gross constants F_1 and F_2 where

$$F_n = \int_0^L \frac{x^n}{r(x)}\,dx, \quad n = 1, 2.$$

We shall show that if two beams with densities $\rho^{(1)}(x^{(1)})$ and $\rho^{(2)}(x^{(2)})$ and rigidities $r^{(1)}(x^{(1)})$ and $r^{(2)}(x^{(2)})$, where $x^{(1)} \in (0, L^{(1)})$ and $x^{(2)} \in (0, L^{(2)})$ have the same impulse response, then

$$[\rho^{(1)}(x^{(1)}(\xi))]^{\frac{3}{8}}[r^{(1)}(x^{(1)}(\xi))]^{\frac{1}{8}}\Psi_n^{(1)}(x^{(1)}(\xi))/\|\Psi_n^{(1)}\|$$
$$= [\rho^{(2)}(x^{(2)}(\xi))]^{\frac{3}{8}}[r^{(2)}(x^{(2)}(\xi))]^{\frac{1}{8}}\Psi_n^{(2)}(x^{(2)}(\xi))/\|\Psi_n^{(2)}\|, \quad n = 1, 2, \ldots,$$

where $$\xi = \xi^{(1)}(x^{(1)}) \equiv \int_0^{x^{(1)}} \left[\frac{\rho^{(1)}(t)}{r^{(1)}(t)}\right]^{\frac{1}{4}} dt$$

and $$\xi = \xi^{(2)}(x^{(2)}) \equiv \int_0^{x^{(2)}} \left[\frac{\rho^{(2)}(t)}{r^{(2)}(t)}\right]^{\frac{1}{4}} dt.$$

As a consequence of these identities, we shall deduce that if $L^{(1)} \leqslant L^{(2)}$, then

$$\left.\begin{aligned}\rho^{(1)}(x) &= \rho^{(2)}(x)\\ r^{(1)}(x) &= r^{(2)}(x)\end{aligned}\right\} \quad \text{for} \quad x \in (0, L^{(1)}),$$

and $$\rho^{(2)}(x) = 1/r^{(2)}(x) = 0 \quad \text{for} \quad x \in (L^{(1)}, L^{(2)}).$$

In other words, the impulse response determines a class of beams. The beams in this class have the same massive front end, and a weightless, infinitely rigid back end of arbitrary length. There is a unique non-singular beam without any back end. The actual construction of this beam is presented in §3. This construction is best made by dealing not with the original equation governing the vibrating beam, but with the auxiliary set of equations:

$$I' = r^{-1}Y, \quad Y' = 2\Theta, \quad \Theta' = K,$$

$$K' = r^{-1}D - \omega^2\rho I, \quad D' = -\omega^2\rho Y.$$

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The auxiliary fields I, Y, Θ, K and D are closely related to the data $\{\omega_n\}$, $\{\nu_n\}$, $\{\mu_n\}$, F_1 and F_2. In particular,

$$Y(0,\omega^2) = -r^2(L)\,F_2 \prod_{n=1}^{\infty}\left(1-\frac{\omega^2}{\mu_n^2}\right),$$

$$\Theta(0,\omega^2) = r^2(L)\,F_1 \prod_{n=1}^{\infty}\left(1-\frac{\omega^2}{\nu_n^2}\right),$$

$$D(0,\omega^2) = r^2(L) \prod_{n=1}^{\infty}\left(1-\frac{\omega^2}{\omega_n^2}\right).$$

We shall see that $I(0,\omega^2)$ and $K(0,\omega^2)$ can also be expressed in terms of the impulse response data. By exploiting the dependence of these fields on ω^2, we shall be able to infer $\rho(x)$ and $r(x)$. This dependence on ω^2 is polynomial for discrete beams of the form

$$\rho(x) = \sum_1^N m_i\,\delta(x-x_i), \quad \frac{1}{r(x)} = \sum_1^N f_i\,\delta(x-x_i),$$

i.e. for beams made up of N clothes-pins of mass $\{m_i\}_1^N$ and stiffness $\{f_i^{-1}\}_1^N$, connected by weightless, infinitely rigid rods of lengths $\{l_i\}_0^{N+1}$, where

$$l_i = x_{i+1}-x_i.$$

We take advantage of these results by first approximating our unknown beam by a discrete one with $N-1$ degrees of freedom for which $\{m_i\}$, $\{f_i\}$ and $\{l_i\}$ are easily found, and then by considering the limit $N\to\infty$. This construction procedure fails if the data are not bona fide. We shall derive some necessary conditions that bona fide data must satisfy and discuss the need for others.

Notation

f	forcing; flaccidity of clothes-pin (discrete beam)
h	arbitrary function in eigenfunction expansion
k, k^*	constants of proportionality between eigenfunctions
l	separation of clothes-pin (discrete beam)
m	mass of clothes-pin (discrete beam); index
r	flexural rigidity
s	Laplace transform variable
t	time, dummy variable of integration
u, v	fundamental solutions of equation for vibrating beam
x	coordinate
y	generic displacement of beam; modal shape
D	auxiliary function associated with free–clamped configuration
F_0, F_1, F_2	zero, first and second moment of flaccidity
G	Green function
I	auxiliary function associated with clamped–clamped configuration
J	auxiliary function associated with non-self-adjoint–clamped configuration
K	auxiliary function associated with Rayleigh–clamped configuration
L	length of beam
N	Green function for non-self-adjoint eigenvalue problem

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U_n	eigenfunctions of canonical fourth-order problem
W	Wronskian
X	$\frac{s^{\frac{1}{4}}}{2^{\frac{1}{2}}}\int_0^x \left(\frac{\rho}{r}\right)^{\frac{1}{4}} dx$, W.K.B.J. variable
Y	auxiliary variable associated with supported–clamped configuration
Z	dimensionless variable for homogeneous beam
η	forcing; eigenfunctions in (3.7)
θ	slope of centre line
λ_n	eigenfrequencies of clamped–clamped beam
μ_n	eigenfrequencies of supported–clamped beam
ν_n	eigenfrequencies of non-self-adjoint–clamped beam
ξ	$\int_0^x \left(\frac{\rho}{r}\right)^{\frac{1}{4}} dx$, coordinate in canonical fourth-order problem
ϖ	$X(L)$
ρ	density
σ_n	eigenfrequencies of Rayleigh–clamped beam
τ	stress
ϕ	fundamental solution of equation for vibrating beam
χ	moment
ψ	fundamental solution of equation for vibrating beam
ω; ω_n	frequency; eigenfrequencies of free–clamped beam
Ξ	$\int_0^L \left(\frac{\rho}{r}\right)^{\frac{1}{4}} dx$
Θ	auxiliary function for non-self-adjoint–clamped configuration
Λ_n	eigenvalues in (3.5)
Φ, Ψ	fundamental solutions: linear combinations of u, v, ϕ and ψ
Ω	Green function for free–clamped beam

1. Preliminary results

We shall be concerned with various solutions of the equation

$$\frac{d^2}{dx^2}\left[r(x)\,\frac{d^2y}{dx^2}\right] = \omega^2\rho(x)\,y, \quad 0 < x < L. \tag{1.1}$$

The flexural rigidity $r(x)$ and the density $\rho(x)$ are non-negative functions, i.e.

$$r(x) \geqslant 0, \quad \rho(x) \geqslant 0.$$

In the section dealing with the uniqueness of the inverse eigenvalue problem, we shall assume that $r(x)$ and $\rho(x)$ are differentiable and in fact that they possess four derivatives. These requirements are very restrictive and can be relaxed in places.

We also shall find it convenient to write (1.1) as a system of first-order differential equations, namely

$$\frac{dy}{dx} = \theta, \quad \frac{d\theta}{dx} = \frac{1}{r}\,\tau, \quad \frac{d\tau}{dx} = -\chi, \quad \frac{d\chi}{dx} = -\omega^2\rho y. \tag{1.2}$$

The variables θ, τ and χ have a simple physical meaning: θ is the slope of the central line, τ is the stress and χ the moment.

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Several fundamental solutions of (1.1) will be useful in the sequel. In particular, we shall define $\phi(x, \omega^2)$ and $\psi(x, \omega^2)$ thus:

$$\phi(0, \omega^2)-1 = \phi'(0, \omega^2) = \phi''(0, \omega^2) = \phi'''(0, \omega^2) = 0, \tag{1.3a}$$

$$\psi(0, \omega^2) = \psi'(0, \omega^2)-1 = \psi''(0, \omega^2) = \psi'''(0, \omega^2) = 0, \tag{1.3b}$$

where primes denote derivatives with respect to x. The functions $\phi(x, \omega^2)$ and $\psi(x, \omega^2)$ are entire functions of ω^2 of order $\frac{1}{4}$. (See Boas (1954, p. 8) for the definition of 'order'.) To prove this assertion one would show (i) that the Taylor series expansions of ϕ and ψ are convergent throughout the entire complex ω^2-plane and (ii) that ϕ and ψ are dominated by entire functions of ω^2 of order $\frac{1}{4}$. The reader is referred to Titchmarsh (1962, pp. 6–11) where a similar proof is given for the Sturm–Liouville equation.

The growth of ϕ and ψ can also be seen from their asymptotic behaviour obtained by the W.K.B.J. method:

$$\phi(x, -s) \sim [\rho^3(0)\,r(0)/\rho^3(x)\,r(x)]^{\frac{1}{8}} \cos X \cosh X, \tag{1.4a}$$

$$\psi(x, -s) \sim \frac{1}{2^{\frac{1}{2}}s^{\frac{1}{4}}}\left[\frac{\rho(0)\,r^3(0)}{\rho^3(x)\,r(x)}\right]^{\frac{1}{8}} (\cos X \sinh X + \sin X \cosh X), \quad |s| \to \infty, \tag{1.4b}$$

where
$$X = \frac{s^{\frac{1}{4}}}{2^{\frac{1}{2}}} \int_0^x \left(\frac{\rho}{r}\right)^{\frac{1}{4}} \mathrm{d}x. \tag{1.5}$$

By replacing s by $-\omega^2$, we see that the entire functions ϕ and ψ are indeed of order $\frac{1}{4}$.

The functions ϕ and ψ are particularly well suited for studying the vibrations of a beam with a free left end, i.e. a beam satisfying the boundary conditions

$$y'' = (ry'')' = 0 \quad \text{at} \quad x = 0. \tag{1.6}$$

In so far as the right end is concerned, we shall deal exclusively with the clamped case, i.e.

$$y = y' = 0 \quad \text{at} \quad x = L. \tag{1.7a}$$

To that effect, we introduce two additional fundamental solutions of (1.1) u and v, say, such that

$$u(L, \omega^2) = u'(L, \omega^2) = u''(L, \omega^2)-1 = u'''(L, \omega^2) = 0, \tag{1.7b}$$

$$v(L, \omega^2) = v'(L, \omega^2) = v''(L, \omega^2) = v'''(L, \omega^2)-1 = 0. \tag{1.7c}$$

Needless to say, u and v are also entire functions of ω^2 of order $\frac{1}{4}$.

The Wronskian $W(x, \omega^2)$,

$$W(x, \omega^2) = \begin{vmatrix} \phi & \psi & u & v \\ \phi' & \psi' & u' & v' \\ \phi'' & \psi'' & u'' & v'' \\ \phi''' & \psi''' & u''' & v''' \end{vmatrix}, \tag{1.8}$$

satisfies the following differential equation:

$$\mathrm{d}[r^2(x)\,W(x, \omega^2)]/\mathrm{d}x = 0. \tag{1.9}$$

Consequently
$$r^2(x)\,W(x, \omega^2) = r^2(0)\,W(0, \omega^2), \tag{1.10a}$$

$$= r^2(L)\,W(L, \omega^2). \tag{1.10b}$$

But
$$W(0, \omega^2) = u''(0, \omega^2)\,v'''(0, \omega^2) - u'''(0, \omega^2)\,v''(0, \omega^2), \tag{1.10c}$$

and
$$W(L, \omega^2) = \phi(L, \omega^2)\,\psi'(L, \omega^2) - \phi'(L, \omega^2)\,\psi(L, \omega^2). \tag{1.10d}$$

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Therefore the Wronskian $W(x, \omega^2)$ vanishes whenever ω coincides with the eigenfrequency ω_n of the beam in the free–clamped configuration.

The eigenvalue problem

$$\left.\begin{aligned} (ry_n'')'' &= \omega_n^2 \rho y_n, \\ y_n'' = (ry_n'')' &= 0 \quad \text{at} \quad x = 0, \\ y_n = y_n' &= 0 \quad \text{at} \quad x = L, \end{aligned}\right\} \tag{1.11}$$

will be the central eigenvalue problem of the paper. However, other eigenvalue problems will occasionally be needed. They differ from (1.11) only in so far as the left boundary conditions are concerned: the right end is always as in (1.11), i.e. clamped. For instance, if the left end were clamped, then we would have:

$$\left.\begin{aligned} (ry_n'')'' &= \lambda_n^2 \rho y_n, \\ y_n = y_n' &= 0 \quad \text{at} \quad x = 0, \\ y_n = y_n' &= 0 \quad \text{at} \quad x = L. \end{aligned}\right\} \tag{1.12}$$

Of course the eigenfunctions of (1.12) are different from those of (1.11). Nevertheless, we use the same notation, namely y_n, to denote both eigenfunctions since no confusion will arise. In fact, we shall only make use of the eigenfrequencies $\{\lambda_n\}$ and not of the eigenfunctions.

The other eigenvalue problems needed in the sequel are those associated with the supported–clamped configuration:

$$\left.\begin{aligned} (ry_n'')'' &= \mu_n^2 \rho y_n, \\ y_n = y_n'' &= 0 \quad \text{at} \quad x = 0, \\ y_n = y_n' &= 0 \quad \text{at} \quad x = L, \end{aligned}\right\} \tag{1.13}$$

and the Rayleigh†–clamped configuration:

$$\left.\begin{aligned} (ry_n'')'' &= \sigma_n^2 \rho y_n, \\ y_n' = (ry_n'')' &= 0 \quad \text{at} \quad x = 0, \\ y_n = y_n' &= 0 \quad \text{at} \quad x = L. \end{aligned}\right\} \tag{1.14}$$

The eigenvalue problems (1.11)–(1.14) are self-adjoint and physically meaningful. The next pair of eigenvalue problems are neither. However, they do enter into our analysis:

$$\left.\begin{aligned} (ry_n'')'' &= \nu_n^2 \rho y_n, \\ y_n = (ry_n'')' &= 0 \quad \text{at} \quad x = 0, \\ y_n = y_n' &= 0 \quad \text{at} \quad x = L; \end{aligned}\right\} \tag{1.15a}$$

$$\left.\begin{aligned} (ry_n'')'' &= \nu_n^2 \rho y_n, \\ y_n' = y_n'' &= 0 \quad \text{at} \quad x = 0, \\ y_n = y_n' &= 0 \quad \text{at} \quad x = L. \end{aligned}\right\} \tag{1.15b}$$

Equation (1.15*b*) is the adjoint of (1.15*a*); hence they have the same eigenfrequencies.

More general boundary conditions could have been considered, but for the sake of presentation, I have restricted my considerations to the aforementioned simple configurations.

† In honour of Lord Rayleigh who touched upon it in his theory of sound (Rayleigh 1945, p. 259). This boundary condition is also referred to as 'sliding'.

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Table 1 summarizes the notation.

TABLE 1

left boundary condition	eigenfrequencies	configuration
$y = y' = 0$	λ_n	clamped
$y = y'' = 0$	μ_n	supported
$y = (ry'')' = 0$	ν_n	non-self-adjoint
$y' = y'' = 0$	ν_n	non-self-adjoint
$y' = (ry'')' = 0$	σ_n	Rayleigh
$y'' = (ry'')' = 0$	ω_n	free

Let us return to our fundamental solutions ϕ, ψ, u and v and record their values for $\omega^2 = 0$:

$$\phi(x, 0) = 1, \tag{1.16a}$$

$$\psi(x, 0) = x, \tag{1.16b}$$

$$u(x, 0) = r'(L)\int_L^x \frac{(x-x')(x'-L)}{r(x')}\,\mathrm{d}x' + r(L)\int_L^x \frac{x-x'}{r(x')}\,\mathrm{d}x', \tag{1.16c}$$

$$v(x, 0) = r(L)\int_L^x \frac{(x-x')(x'-L)}{r(x')}\,\mathrm{d}x'. \tag{1.16d}$$

As a result, we deduce from (1.10d) that

$$W(L, 0) = 1. \tag{1.17}$$

Making use of Hadamard's factorization theorem for entire functions (Boas 1954, p. 22) for $W(L, \omega^2)$, which is also an entire function of order $\frac{1}{4}$, we can write

$$W(L, \omega^2) = W(L, 0)\prod_{m=1}^{\infty}\left(1-\frac{\omega^2}{\omega_m^2}\right) \tag{1.18a}$$

or, on account of (1.17),

$$W(L, \omega^2) = \prod_{m=1}^{\infty}\left(1-\frac{\omega^2}{\omega_m^2}\right). \tag{1.18b}$$

The asymptotic behaviour of the eigenfrequencies $\{\omega_n\}$ for n large, which we shall discuss presently, guarantees the convergence of the infinite product. Inserting (1.18a) in (1.10b), we get

$$W(x, \omega^2) = \left[\frac{r(L)}{r(x)}\right]^2 \prod_{m}^{\infty}\left(1-\frac{\omega^2}{\omega_m^2}\right). \tag{1.19}$$

The fundamental solutions u and v will not be used in their present form. Rather they will enable us to define two new fundamental solutions:

$$\Phi(x, \omega^2) = r(x)\begin{vmatrix} \psi & u & v \\ \psi' & u' & v' \\ \psi'' & u'' & v'' \end{vmatrix} \tag{1.20}$$

and

$$\Psi(x, \omega^2) = r(x)\begin{vmatrix} \phi & u & v \\ \phi' & u' & v' \\ \phi'' & u'' & v'' \end{vmatrix}. \tag{1.21}$$

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It is simple to check that Φ and Ψ are indeed solutions of (1) as well as that

$$\Phi''(0, \omega^2) = \Phi(L, \omega^2) = \Phi'(L, \omega^2) = 0, \tag{1.22}$$

$$(r\Psi'')'(0, \omega^2) = \Psi(L, \omega^2) = \Psi'(L, \omega^2) = 0. \tag{1.23}$$

The asymptotic behaviour of Φ and Ψ obtained once again by means of a W.K.B.J. approach is

$$\Phi(x, -s) \sim \frac{1}{2^{\frac{1}{2}}s^{\frac{3}{4}}} \frac{r^{\frac{3}{2}}(L)}{\rho^{\frac{1}{2}}(L)} \left[\frac{r^3(0)\,\rho(0)}{r(x)\,\rho^3(x)}\right]^{\frac{1}{8}} \{(\cos\varpi \sinh\varpi + \sin\varpi\cosh\varpi)\sin(X-\varpi)\sinh(X-\varpi)$$
$$+\cos\varpi\cosh\varpi\,[\sin(X-\varpi)\cosh(X-\varpi) - \cos(X-\varpi)\sinh(X-\varpi)]\}, \tag{1.24a}$$

$$\Psi(x, -s) \sim \frac{1}{2s^{\frac{1}{2}}} \frac{r^{\frac{3}{2}}(L)}{\rho^{\frac{1}{2}}(L)} \left[\frac{r(0)\,\rho^3(0)}{r(x)\,\rho^3(x)}\right]^{\frac{1}{8}} \{2\cos\varpi\cosh\varpi\sin(X-\varpi)\sinh(X-\varpi) + (\cos\varpi\sinh\varpi$$
$$-\sin\varpi\cosh\varpi)\,[\sin(X-\varpi)\cosh(X-\varpi) - \cos(X-\varpi)\sinh(X-\varpi)]\}, \tag{1.24b}$$

where

$$\varpi = \frac{s^{\frac{1}{4}}}{2^{\frac{1}{2}}} \int_0^L \left(\frac{\rho}{r}\right)^{\frac{1}{4}} \mathrm{d}x,$$

and X is given in (1.5). Finally, by substituting (1.4a) and (1.4b) in (1.10d) we can deduce that

$$r^2(0)\,W(0, -s) \sim \tfrac{1}{2}\{[\rho(0)/\rho(L)]\,r(0)\,r^3(L)\}^{\frac{1}{2}}(\cos^2\varpi + \cosh^2\varpi). \tag{1.25}$$

In view of (1.22), we can use the function Φ to solve the eigenvalue problems (1.11), (1.13) and (1.15b). In particular, from (1.13) we deduce that

$$\Phi(0, \omega^2) = \Phi(0, 0) \prod_m^\infty \left(1 - \frac{\omega^2}{\mu_m^2}\right)$$

and, making use of the formulas (1.16) which express the behaviour near $x = 0$, we see that

$$\Phi(0, \omega^2) = r^2(L)\,F_2 \prod_m^\infty \left(1 - \frac{\omega^2}{\mu_m^2}\right), \tag{1.26}$$

where F_2 is the second moment of the 'flaccidity':

$$F_2 = \int_0^L \frac{x^2}{r(x)}\,\mathrm{d}x. \tag{1.27}$$

Similarly, in view of (1.23) and (1.15a) we can write

$$\Psi(0, \omega^2) = \Psi(0, 0) \prod_m^\infty \left(1 - \frac{\omega^2}{\nu_m^2}\right),$$

and on account of (1.16)

$$\Psi(0, \omega^2) = r^2(L)\,F_1 \prod_m^\infty \left(1 - \frac{\omega^2}{\nu_m^2}\right), \tag{1.28}$$

where F_1 is the first moment of the flaccidity:

$$F_1 = \int_0^L \frac{x}{r(x)}\,\mathrm{d}x. \tag{1.29}$$

Finally, by similar means we can show that

$$\Phi'(0, \omega^2) = -r^2(L)\,F_1 \prod_m^\infty \left(1 - \frac{\omega^2}{\nu_m^2}\right), \tag{1.30a}$$

$$r(0)\,\Phi'''(0, \omega^2) = r^2(L) \prod_m^\infty \left(1 - \frac{\omega^2}{\omega_m^2}\right). \tag{1.30b}$$

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The functions Φ and Ψ are linearly independent as long as ω differs from an eigenvalue ω_n of the beam in the free–clamped configuration. In this case we have

$$\Phi(x, \omega_n^2) = k_n \Psi(x, \omega_n^2).\dagger \tag{1.31}$$

The constants of proportionality, k_n, can be deduced by considering (1.31) with $x = 0$ and using (1.26) and (1.28):

$$k_n = \frac{F_2}{F_1} \prod_m^\infty \frac{1-\omega_n^2/\mu_m^2}{1-\omega_n^2/\nu_m^2}. \tag{1.32}$$

To simplify the notation, we shall define

$$\Psi_n(x) \equiv \Psi(x, \omega_n^2). \tag{1.33}$$

Thus $\{\Psi_n, \omega_n\}$ are the eigensolutions of (1.11). We can also represent the eigenfunctions of (1.11) in terms of ϕ and ψ. Thus we have another formula akin to (1.31):

$$[\psi(L, \omega_n^2)/\phi(L, \omega_n^2)]\,\phi(x, \omega_n^2) - \psi(x, \omega_n^2) = k_n^* \Psi_n(x). \tag{1.34}$$

It should be noted that from the definitions (1.20) and (1.21) of Φ and Ψ we can deduce that

$$\Phi''(L, \omega^2) = r(L)\,\psi(L, \omega^2),$$

$$\Psi''(L, \omega^2) = r(L)\,\phi(L, \omega^2).$$

As a result

$$\frac{\psi(L, \omega_n^2)}{\phi(L, \omega_n^2)} = \frac{\Phi''(L, \omega_n^2)}{\Psi''(L, \omega_n^2)} = k_n \tag{1.35}$$

and so (1.34) can be written as follows:

$$k_n \phi(x, \omega_n^2) - \psi(x, \omega_n^2) = k_n^* \Psi_n(x). \tag{1.34'}$$

To find the values of k_n^*, we set $x = 0$ and use (1.28):

$$k_n^* = k_n \Big/ r^2(L)\, F_1 \prod_m^\infty \left(1 - \frac{\omega_n^2}{\nu_m^2}\right). \tag{1.36}$$

We consider next the asymptotic form of the eigenfrequencies of the five problems. It is easy to see that

$$\Phi(0, \lambda_m^2)\,\Psi'(0, \lambda_m^2) - \Phi'(0, \lambda_m^2)\,\Psi(0, \lambda_m^2) = 0, \tag{1.37a}$$

$$\Phi(0, \mu_m^2) = 0, \tag{1.37b}$$

$$\Psi(0, \nu_m^2) = \Phi'(0, \nu_m^2) = 0, \tag{1.37c}$$

$$\Psi'(0, \sigma_m^2) = 0, \tag{1.37d}$$

$$(r\Phi'')'(0, \omega_m^2) = \Psi''(0, \omega_m^2) = 0. \tag{1.37e}$$

In fact (1.37*b*) and (1.37*c*) are equivalent to (1.26), (1.28) and (1.30). By replacing Φ and Ψ in the above equations by their asymptotic forms as given in (1.24*a*) and (1.24*b*) we deduce that

$$\left.\begin{aligned} \lambda_n^2 \sim (n+\tfrac{1}{2})^4 \pi^4/\Xi^4, \quad \mu_n^2 \sim (n+\tfrac{1}{4})^4 \pi^4/\Xi^4, \\ \nu_n^2 \sim n^4 \pi^4/\Xi^4, \\ \sigma_n^2 \sim (n-\tfrac{1}{4})^4 \pi^4/\Xi^4, \quad \omega_n^2 \sim (n-\tfrac{1}{2})^4 \pi^4/\Xi^4, \end{aligned}\right\} \tag{1.38}$$

† We shall see in §3 that the sequence $\{\omega_n\}$ is distinct from $\{\mu_n\}$ and $\{\nu_n\}$. As a result, $\Phi(x, \omega_n^2)$ and $\psi(x, \omega_n^2)$ cannot be identically zero. This conclusion can also be reached by appealing to results in Leighton & Nehari (1958). I am indebted to E. Trubowitz for bringing this fact to my attention.

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where $$\Xi = \int_0^L \left(\frac{\rho}{r}\right)^{\frac{1}{4}} \mathrm{d}x. \tag{1.39}$$

Other important properties of the eigenvalues will be needed. For instance, we shall exploit the fact that λ_n, μ_n, ν_n, σ_n and ω_n are all simple eigenvalues. This result is a consequence of the theory of integral equations with oscillating kernels (Gantmakher & Krein, 1950). We can only give the briefest outline here. The first step is to convert the eigenvalue problems (1.11)–(1.15) into integral equations of the form

$$y(x) = \omega^2 \int_0^L G(x, t)\, \rho(t)\, y(t)\, \mathrm{d}t. \tag{1.40}$$

If the kernels $G(x, t)$, which are the Green functions for the various problems, satisfy the following properties:

(i) $$G\begin{pmatrix} x_1 \dots x_n \\ t_1 \dots t_n \end{pmatrix} \equiv \begin{vmatrix} G(x_1, t_1) & \dots & G(x_1, t_n) \\ \vdots & & \vdots \\ G(x_n, t) & \dots & G(x_n, t_n) \end{vmatrix} \geqslant 0 \tag{1.41}$$

for all partitions

$$0 < \begin{pmatrix} x_1 < x_2 < \dots < x_n \\ t_1 < t_2 < \dots < t_n \end{pmatrix} < L, \quad n = 1, 2, \dots;$$

(ii) $$G\begin{pmatrix} x_1 \dots x_n \\ x_1 \dots x_n \end{pmatrix} > 0 \tag{1.42}$$

for all partitions and all values of n;

(iii) $$G(x, t) > 0 \quad \text{for} \quad 0 < x, t < L, \tag{1.43}$$

then the kernel $G(x, t)$ is said to be oscillating. For such kernels, Gantmakher (1936) (see also Gantmakher & Krein 1950) showed that the eigenvalues of (1.40) are simple.

Gantmakher & Krein (1950) have shown that the Green functions for four out of the five eigenvalue problems of interest to us are indeed oscillating kernels. These are the four self-adjoint problems (1.11)–(1.14); thus λ_n, μ_n, σ_n and ω_n are simple eigenvalues.

The non-self-adjoint problem (1.15*a*) has not been considered. However, by means of a theorem of Karlin (1971), we can state that the Green function for that problem satisfies (1.41), i.e. is 'totally positive' in Karlin's parlance. The fact that the kernel of the integral equation is a 'totally positive' Green function implies that (1.42) is also satisfied. The proof of this assertion can be found in Krein & Finkelstein (1939). Thus we only need to prove that (1.43) holds.

It is simple to check that the Green function under consideration denoted by $N(x, t)$ is given by

$$N(x, t) = \Omega(x, t) - \frac{1}{F_1} \int_x^L \frac{z-x}{r(z)}\, \mathrm{d}z \int_t^L \frac{z(z-t)}{r(z)}\, \mathrm{d}z, \tag{1.44}$$

where $$\Omega(x, t) = \begin{cases} \displaystyle\int_x^L \frac{(z-x)(z-t)}{r(z)}\, \mathrm{d}z, & x \geqslant t, \\ \displaystyle\int_t^L \frac{(z-x)(z-t)}{r(z)}\, \mathrm{d}z, & x \leqslant t. \end{cases} \tag{1.45}$$

Consequently,

$$N(t, t) = \int_t^L \frac{(z-t)^2}{r(z)}\, \mathrm{d}z - \frac{1}{F_1} \int_t^L \frac{z-t}{r(z)}\, \mathrm{d}z \int_t^L \frac{z(z-t)}{r(z)}\, \mathrm{d}z$$

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or

$$N(t,t) = \left[1-\left(\int_t^L \frac{z\,dz}{r(z)}\Big/F_1\right)\right]\int_t^L \frac{(z-t)^2}{r(z)}\,dz+\frac{t}{F_1}\left\{\int_t^L \frac{z^2dz}{r(z)}\int_t^L \frac{dz}{r(z)}-\left[\int_t^L \frac{z\,dz}{r(z)}\right]^2\right\} \geqslant 0. \quad (1.46)$$

Now, over the interval $(0, t)$

$$r(x)\frac{\partial^2 N}{\partial x^2} = -\frac{1}{F_1}\int_t^L \frac{(z-t)z}{r(z)}\,dz \leqslant 0 \quad (1.47)$$

and hence N is concave downward, which together with (1.46) implies that

$$N(x,t) \geqslant 0 \quad \text{for} \quad 0 \leqslant x \leqslant t.$$

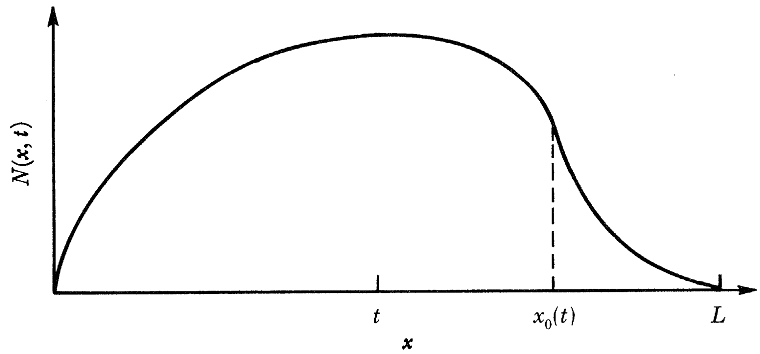

FIGURE 1. Sketch of the Green function $N(x, t)$ for the non-self-adjoint–clamped configuration. The curvature of $N(x, t)$ changes at $x = x_0(t)$.

Over the remaining interval (t, L),

$$r(x)\frac{\partial^2 N}{\partial x^2} = x-t-\frac{1}{F_1}\int_t^L \frac{(z-t)z}{r(z)}\,dz$$

or

$$r(x)\frac{\partial^2 N}{\partial x^2} = x-x_0(t), \quad (1.48)$$

where

$$x_0(t) = \frac{1}{F_1}\left[t\int_0^t \frac{z\,dz}{r(z)}+\int_t^L \frac{z^2dz}{r(z)}\right]. \quad (1.49)$$

Clearly

$$t \leqslant x_0(t) \leqslant L$$

and so $N(x, t)$ is respectively concave downward and upward over $(t, x_0(t))$ and $(x_0(t), L)$. Note that

$$N(x_0(t),t) = \Omega(x_0(t),t)-[x_0(t)-t]\int_{x_0(t)}^L \frac{z-x_0(t)}{r(z)}\,dz,$$

i.e.

$$N(x_0(t),t) = \int_{x_0(t)}^L \frac{(z-x_0)^2}{r(z)}\,dz \geqslant 0$$

and so

$$N(x,t) \geqslant 0 \quad \text{for} \quad 0 \leqslant x \leqslant x_0(t)$$

(see figure 1). We obtain the desired result, by noting that over the interval $(x_0(t), L)$, $N(x, t)$ must lie above its tangents. The x-axis being one such tangent, N must therefore be positive in this interval. Consequently

$$N(x,t) > 0 \quad \text{for} \quad 0 < x, t < L,$$

i.e. $N(x, t)$ is an oscillating kernel and hence the ν_n are simple eigenvalues.

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2. Impulse response and uniqueness of the inverse problem

2.1. *The impulse response*

Let us consider the following artificial problem

$$\frac{\partial^2}{\partial x^2}\left[r(x)\frac{\partial^2 \hat{y}}{\partial x^2}\right] = -\rho(x)\frac{\partial \hat{y}}{\partial t}, \quad 0 < x < L, t > 0, \tag{2.1}$$

with

$$\left.\begin{aligned} \frac{\partial^2 \hat{y}(0,t)}{\partial x^2} &= \frac{\partial}{\partial x}\left[r(x)\frac{\partial^2 \hat{y}(0,t)}{\partial x^2}\right] = 0, \\ \hat{y}(L,t) &= \frac{\partial \hat{y}(L,t)}{\partial x} = 0, \\ \hat{y}(x,0) &= \eta(x). \end{aligned}\right\} \tag{2.2}$$

After making a Laplace transform where

$$y(x,-s) = \int_0^\infty e^{-st}\hat{y}(x,t)\,dt,$$

we can write (2.1) as follows:

$$(ry'')'' + \rho s y = \rho\eta \tag{2.3}$$

with

$$\left.\begin{aligned} y'' = (ry'')' &= 0 \quad \text{at} \quad x = 0, \\ y = y' &= 0 \quad \text{at} \quad x = L. \end{aligned}\right\} \tag{2.4}$$

To solve the above boundary value problem, we introduce the appropriate Green function

$$\left.\begin{aligned} (r\Omega'')'' + \rho s\Omega &= \delta(x-z), \\ \Omega'' = (r\Omega'')' &= 0 \quad \text{at} \quad x = 0, \\ \Omega = \Omega' &= 0 \quad \text{at} \quad x = L, \end{aligned}\right\} \tag{2.5}$$

which enables us to write

$$y(x,-s) = \int_0^L \Omega(x,z;s)\rho(z)\eta(z)\,dz. \tag{2.6}$$

If we were to replace s by $-\omega^2$, then we would be back to our beam which is here excited by a force $\eta(x)$ applied at time $t = 0$. The case for which

$$\eta(x) = \delta(x)/\rho(0),$$

would then correspond to a point force applied at the free left end at time $t = 0$. By impulse response, we shall mean the measurements of the displacement and slope of the centre line at that left end, i.e.

$$y(0,-s) = \Omega(0,0;s), \quad \theta(0,-s) = \partial\Omega(0,0;s)/\partial x. \tag{2.7}$$

Now, making use of the functions ϕ, ψ, Φ and Ψ introduced previously, we can check that

$$\Omega(x,z;-s) = \begin{cases} \dfrac{\phi(x,-s)\,\Phi(z,-s) - \psi(x,-s)\,\Psi(z,-s)}{r^2(0)\,W(0,-s)}, & x \leqslant z, \\ \dfrac{\phi(z,-s)\,\Phi(x,-s) - \psi(z,-s)\,\Psi(x,-s)}{r^2(0)\,W(0,-s)}, & x \leqslant z. \end{cases} \tag{2.8}$$

Consequently

$$y(0,-s) = \frac{1}{r^2(0)}\frac{\Phi(0,-s)}{W(0,-s)}$$

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and
$$\theta(0, -s) = -\frac{1}{r^2(0)} \frac{\Psi(0, -s)}{W(0, -s)},$$

or, in view of (1.19), (1.26) and (1.28),

$$y(0, -s) = F_2 \prod_{m=1}^{\infty} \frac{1+s/\mu_m^2}{1+s/\omega_m^2}, \tag{2.9a}$$

$$\theta(0, -s) = -F_1 \prod_{m=1}^{\infty} \frac{1+s/\nu_m^2}{1+s/\omega_m^2}. \tag{2.9b}$$

The impulse response contains the following information:

(i) the first and second moments F_1 and F_2 of the 'flaccidity';

(ii) the three spectra $\{\omega_n\}$, $\{\nu_n\}$ and $\{\mu_n\}$ corresponding respectively to the free, non-self-adjoint and supported boundary conditions at the left end.

We shall discuss next whether the information contained in the impulse response is sufficient to characterize a beam uniquely.

2.2. *Beams with identical spectra*

Let us say that the beam characterized by $r(x)$, $\rho(x)$, of length L, clamped at the right end, has spectra $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{\sigma_n\}$ and $\{\omega_n\}$. There are other distinct beams with the same spectra. Indeed, the two-parameter family of beams characterized by

$$\tilde{r}(\tilde{x}) = \alpha r((\alpha/\beta)^{\frac{1}{4}}x), \quad \tilde{\rho}(\tilde{x}) = \beta\rho((x/\beta)^{\frac{1}{4}}x), \tag{2.10}$$

has the same spectra. (The parameters α and β are positive.) The lengths of these beams are

$$\tilde{L} = (\alpha/\beta)^{\frac{1}{4}} L. \tag{2.11}$$

The first and second moments of the flaccidity of these beams are:

$$\tilde{F}_1 = (1/\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}})\, F_1, \quad \tilde{F}_2 = (1/\alpha^{\frac{1}{4}}\beta^{\frac{3}{4}})\, F_2. \tag{2.12}$$

If we were to impose the additional requirement that these two moments match those of the beam under consideration, then we would obviously see that $\alpha = \beta = 1$. Hence, the role of F_1 and F_2 is to set the scale of the beam being reconstructed.

2.3. *Pathological beams with the same impulse response*

It is possible to construct two distinct beams with the same impulse response?

Consider a beam $\rho(x), r(x)$ of length L, clamped at the right end. Construct a second beam with the following characteristics:

$$\tilde{\rho}(x) = \begin{cases} \rho(x), & 0 < x < L, \\ \bar{\rho}, & L < x < \tilde{L}; \end{cases} \tag{2.13a}$$

$$\tilde{r}(x) = \begin{cases} r(x), & 0 < x < L, \\ \bar{r}, & L < x < \tilde{L}. \end{cases} \tag{2.13b}$$

In other words, the second beam is identical to the first over the interval $(0, L)$ and then has constant properties over the section $(L, \tilde{L})$. Abusing our notation, let us denote by $\phi(x, \omega^2)$, $\psi(x, \omega^2)$ two fundamental solutions of (1.1) that satisfy the boundary conditions at $x = 0$ appropriate for either one of the three configurations entering into the impulse response.

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The solution over the interval $(L, \tilde{L})$ is written in terms of functions

$$\cos\,[(\bar{\rho}\omega^2/\bar{r})^{\frac{1}{4}}(x-\tilde{L})] - \cosh\,[(\bar{\rho}\omega^2/\bar{r})^{\frac{1}{4}}(x-\tilde{L})]$$

and

$$\sin\,[(\bar{\rho}\omega^2/\bar{r})^{\frac{1}{4}}(x-\tilde{L})] - \sinh\,[(\bar{\rho}\omega^2/\bar{r})^{\frac{1}{4}}(x-\tilde{L})]$$

which satisfy the clamped conditions identically.

Piecing the solutions in $(0, L)$ and $(L, \tilde{L})$ by requiring that

$$[\tilde{y}] = [\tilde{y}'] = [\tilde{r}\tilde{y}''] = [(\tilde{r}\tilde{y}'')'] = 0 \tag{2.14}$$

where $[.]$ denotes the jump from $L+0$ to $L-0$, we obtain the determinantal equations yielding the various spectra:

$$0 = \begin{vmatrix} \phi(L,\omega^2) & \psi(L,\omega^2) & \cos Z - \cosh Z & \sin Z - \sinh Z \\ \phi'(L,\omega^2) & \psi'(L,\omega^2) & -(\bar{\rho}^{\frac{1}{4}}/\bar{r})\,\omega^{\frac{1}{2}}(\sin Z + \sinh Z) & (\bar{\rho}^{\frac{1}{4}}/\bar{r})\,\omega^{\frac{1}{2}}(\cos Z - \cosh Z) \\ (r\phi'')(L,\omega^2) & (r\psi'')(L,\omega^2) & -\bar{r}^{\frac{1}{2}}\bar{\rho}^{\frac{1}{2}}\,\omega(\cos Z + \cosh Z) & -\bar{r}^{\frac{1}{2}}\bar{\rho}^{\frac{1}{2}}\,\omega(\sin Z + \sinh Z) \\ (r\phi'')'(L,\omega^2) & (r\psi'')'(L,\omega^2) & \bar{r}^{\frac{1}{4}}\bar{\rho}^{\frac{3}{4}}\omega^{\frac{3}{2}}(\sin Z - \sinh Z) & -\bar{r}^{\frac{1}{4}}\bar{\rho}^{\frac{3}{4}}\omega^{\frac{3}{2}}(\cos Z + \cosh Z) \end{vmatrix}, \tag{2.15}$$

where

$$Z = (\bar{\rho}^{\frac{1}{4}}/\bar{r})\,\omega^{\frac{1}{2}}(L-\tilde{L}).$$

Now we consider the limit

$$\bar{\rho}/\bar{r} \to 0 \tag{2.16a}$$

but such that

$$\bar{r}\bar{\rho} = \alpha. \tag{2.16b}$$

This means, that in the limit the added section is made up of a weightless rod of infinite rigidity. Then dividing the last column of (2.15) by $\bar{r}^{\frac{1}{4}}\bar{\rho}^{\frac{3}{4}}$ and taking the limit mentioned above, we note that (2.15) becomes

$$\begin{vmatrix} \phi & \psi & 0 & 0 \\ \phi' & \psi' & 0 & 0 \\ r\phi'' & r\psi' & -2\alpha\omega & -\omega \\ (r\phi'')' & (r\psi'')' & 0 & -2\omega^{\frac{3}{2}} \end{vmatrix} = 0$$

or

$$\begin{vmatrix} \phi & \psi \\ \phi' & \psi' \end{vmatrix} = 0.$$

But this is the generic determinental equation for the smaller beam! In addition, since the flaccidity of the second beam is zero over the additional interval $(L, \tilde{L})$

$$\tilde{F}_1 = F_1, \quad \tilde{F}_2 = F_2.$$

Thus, for the clamped-right-end case, given any beam with impulse response $\{\lambda_n\}$, $\{\nu_n\}$, $\{\omega_n\}$, F_1 and F_2, we can construct a class of beams with the same impulse response, by simply tacking onto the original beam an infinitely rigid weightless rod of arbitrary length.

This result is easy to understand physically: the section $(L, \tilde{L})$, because of its very nature, is never set in motion by the initial excitation and hence never felt.

The question remains whether there are other types of less pathological beams that have the same impulse response. The answer is no.

2.4. *Toward the uniqueness theorem*

We assume that two beams $\rho^{(1)}$, $r^{(1)}$ and $\rho^{(2)}$, $r^{(2)}$ have the same impulse response. Since the information about the length of the beams is not part of the impulse response, we must assume that their lengths, $L^{(1)}$ and $L^{(2)}$, are not necessarily equal. This presents a first obstacle to the

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adaptation of Levinson's (1949) classical proof of the uniqueness of the inverse Sturm–Liouville equation to the problem at hand. Indeed, to compare the two beams we must bring them to some common ground. If $x^{(1)}$ and $x^{(2)}$ are coordinates suitable for each beam, then we can accomplish our aim by introducing a new variable ξ as follows:

$$\xi = \int_0^{x^{(1)}} \left[\frac{\rho^{(1)}}{r^{(1)}}\right]^{\frac{1}{4}} \mathrm{d}x \tag{2.17a}$$

and

$$\xi = \int_0^{x^{(2)}} \left[\frac{\rho^{(2)}}{r^{(2)}}\right]^{\frac{1}{4}} \mathrm{d}x. \tag{2.17b}$$

Note that, as $x^{(1)}$ and $x^{(2)}$ vary from 0 to respectively $L^{(1)}$ and $L^{(2)}$, ξ varies from 0 to Ξ which was defined in (1.39). This is a consequence of the fact that eigenvalues of the two beams have the same asymptotic behaviour.

The functions $\xi(x^{(1)})$ and $\xi(x^{(2)})$ defined in (2.17) are non-decreasing, and hence are invertible:

$$x^{(1)} = x^{(1)}(\xi), \quad x^{(2)} = x^{(2)}(\xi). \tag{2.18}$$

We are now ready to define the following strange integral of a hybrid version of the Green function Ω given in (2.8), namely

$$\mathscr{I}(\xi) = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}s \int_0^{\Xi} \rho^{(2)}(x^{(2)}(\xi))\, \boldsymbol{\Omega}^{(2,1)}(x^{(2)}(\zeta), x^{(1)}(\xi); -s)\, h(\zeta) \frac{\mathrm{d}x^{(2)}}{\mathrm{d}\zeta}\, \mathrm{d}\zeta, \tag{2.19}$$

where

$$\boldsymbol{\Omega}^{(2,1)}(x^{(2)}(\zeta), x^{(1)}(\xi); -s)$$
$$= \begin{cases} \dfrac{\phi^{(2)}(x^{(2)}(\zeta), -s)\, \Phi^{(1)}(x^{(1)}(\xi), -s) - \psi^{(2)}(x^{(2)}(\zeta), -s)\, \Psi^{(1)}(x^{(1)}(\xi), -s)}{r^2(0)\, W^{(1)}(0, -s)}, & x^{(2)} \leqslant x^{(1)}, \\[2ex] \dfrac{\phi^{(1)}(x^{(1)}(\xi), -s)\, \Phi^{(2)}(x^{(2)}(\zeta), -s) - \psi^{(1)}(x^{(1)}(\xi), -s)\, \Psi^{(2)}(x^{(2)}(\zeta), -s)}{r^2(0)\, W^{(2)}(0, -s)}, & x^{(1)} \leqslant x^{(2)}, \end{cases} \tag{2.20}$$

and $h(\zeta)$ is an arbitrary function.

We digress at this stage to explain why $r(0)$ enters into the definition of $\boldsymbol{\Omega}^{(2,1)}$ without any superscript. This is due to the fact that

$$r^{(1)}(0) = r^{(2)}(0). \tag{2.21}$$

This result, as well as

$$\rho^{(1)}(0) = \rho^{(2)}(0), \tag{2.22}$$

will be derived next.

2.5. *End-point identities*

The preceding results are obtained by confronting the product representation (1.26) of $\phi^{(i)}(0, \omega^2)$ with its asymptotic representation (1.24*a*). Indeed, this implies that

$$[r^{(i)}(L)^{(i)}]^2 F_2 \prod_{m=1}^{\infty} \left(1 + \frac{s}{\mu_m^2}\right) \sim \frac{1}{2(2^{\frac{1}{2}})\, s^{\frac{3}{4}}} \frac{[r^{(i)}(L^{(i)})]^{\frac{3}{2}}}{[\rho^{(i)}(L^{(i)})]^{\frac{1}{2}}} \left[\frac{r^{(i)}(0)}{\rho^{(i)}(0)}\right]^{\frac{1}{4}} [\sinh 2^{\frac{1}{2}} s^{\frac{1}{4}} \Xi - \sin 2^{\frac{1}{2}} s^{\frac{1}{4}} \Xi], \tag{2.23}$$

where $i = 1, 2$. Consequently

$$\frac{1}{[r^{(1)}(L^{(1)})]^{\frac{1}{2}}} \frac{1}{[\rho^{(1)}(L^{(1)})]^{\frac{1}{2}}} \left[\frac{r^{(1)}(0)}{\rho^{(1)}(0)}\right]^{\frac{1}{4}} = \frac{1}{[r^{(2)}(L^{(2)})]^{\frac{1}{2}}} \frac{1}{[\rho^{(2)}(L^{(2)})]^{\frac{1}{2}}} \left[\frac{r^{(2)}(0)}{\rho^{(2)}(0)}\right]^{\frac{1}{4}}. \tag{2.24}$$

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The same procedure applied to $\Psi^{(i)}(0, \omega^2)$, namely

$$[r^{(i)}(L^{(i)})]^2 F_1 \prod_{m=1}^{\infty}\left(1+\frac{s}{\nu_m^2}\right) \sim \frac{1}{2s^{\frac{1}{2}}}\frac{[r^{(i)}(L^{(i)})]^{\frac{3}{4}}}{[\rho^{(i)}(L^{(i)})]^{\frac{1}{4}}}[\cosh^2(2^{-\frac{1}{2}}s^{\frac{1}{4}}\Xi)-\cos^2(2^{-\frac{1}{2}}s^{\frac{1}{4}}\Xi)], \tag{2.25}$$

yields
$$\frac{1}{[r^{(1)}(L^{(1)})]^{\frac{1}{4}}}\frac{1}{[\rho^{(1)}(L^{(1)})]^{\frac{1}{4}}} = \frac{1}{[r^{(2)}(L^{(2)})]^{\frac{1}{4}}}\frac{1}{[\rho^{(2)}(L^{(2)})]^{\frac{1}{4}}}. \tag{2.26}$$

In the same vein, we can confront the product representation (1.19) for $W(0, -s)$ with its asymptotic representation (1.26), namely

$$[r^{(i)}(L^{(i)})]^2 \prod_{m=1}^{\infty}\left(1+\frac{s}{\omega_m^2}\right) \sim \frac{1}{2}\left[\frac{\rho^{(i)}(0)\,r^{(i)}(0)}{\rho^{(i)}(L^{(i)})}\right]^{\frac{1}{4}}[r^{(i)}(L^{(i)})]^{\frac{3}{4}}[\cosh^2(2^{-\frac{1}{2}}s^{\frac{1}{4}}\Xi)+\cos^2(2^{-\frac{1}{2}}s^{\frac{1}{4}}\Xi)], \tag{2.27}$$

from which we deduce that

$$\left[\frac{r^{(1)}(0)}{r^{(1)}(L^{(1)})}\right]^{\frac{1}{4}}\left[\frac{\rho^{(1)}(0)}{\rho^{(1)}(L^{(1)})}\right]^{\frac{1}{4}} = \left[\frac{r^{(2)}(0)}{r^{(2)}(L^{(2)})}\right]^{\frac{1}{4}}\left[\frac{\rho^{(2)}(0)}{\rho^{(2)}(L^{(2)})}\right]^{\frac{1}{4}}. \tag{2.28}$$

Making use of (2.26) we can rewrite (2.24) and (2.28) as follows

$$r^{(1)}(0)/\rho^{(1)}(0) = r^{(2)}(0)/\rho^{(2)}(0),$$
$$r^{(1)}(0)\,\rho^{(1)}(0) = r^{(2)}(0)\,\rho^{(2)}(0),$$

which imply (2.21) and (2.22). Therefore, if the two beams clamped at the right end have the same impulse response, their densities and rigidities must coincide at the left end. Therefore, we are justified in omitting the superscripts.

In addition, they must satisfy (2.26), i.e.

$$r^{(1)}(L^{(1)})\,\rho^{(1)}(L^{(1)}) = r^{(2)}(L^{(2)})\,\rho^{(2)}(L^{(2)}).$$

Note that this product is finite even for the pathological beams previously discussed.

2.6. *Evaluation of $\mathscr{I}(\xi)$ by calculus of residues*

An examination of the integrand of $\mathscr{I}(\xi)$ reveals that the path of integration can be closed by a half-circle in the left half of the s-plane. The only singularities are simple poles at $s = -\omega_n^2$, and a straightforward application of the calculus of residues yields

$$\begin{aligned}\mathscr{I}(\xi) = \sum_{n=1}^{\infty}\Bigg\{\Bigg[&\Phi_n^{(1)}(x^{(1)})\int_0^{\xi}\rho^{(2)}(x^{(2)})\,\phi^{(2)}(x^{(2)},\omega_n^2)\,h\,\frac{dx^{(2)}}{d\zeta}\,d\zeta \\ &-\Psi_n^{(1)}(x^{(1)})\int_0^{\xi}\rho^{(2)}(x^{(2)})\,\psi^{(2)}(x^{(2)},\omega^2)\,h\,\frac{dx^{(2)}}{d\zeta}\,d\zeta\Bigg]\,\omega_n^2\Bigg/[r^{(1)}(L^{(1)})]^2\prod_{m\neq n}\left(1-\frac{\omega_n^2}{\omega_m^2}\right) \\ &+\Bigg[\phi^{(1)}(x^{(1)},\omega_n^2)\int_{\xi}^{\Xi}\rho^{(2)}(x^{(2)})\,\Phi_n^{(2)}(x^{(2)})\,h\,\frac{dx^{(2)}}{d\zeta}\,d\zeta \\ &-\psi^{(1)}(x^{(1)},\omega_n^2)\int_{\xi}^{\Xi}\rho^{(2)}(x^{(2)})\,\Psi_n^{(2)}(x^{(2)})\,h\,\frac{dx^{(2)}}{d\zeta}\,d\zeta\Bigg]\,\omega_n^2\Bigg/[r^{(2)}(L^{(2)})]^2\prod_{m\neq n}\left(1-\frac{\omega_n^2}{\omega_m^2}\right)\Bigg\}.\end{aligned} \tag{2.29}$$

In (2.29), $x^{(1)}$ and $x^{(2)}$ are not written explicitly as functions of ξ and ζ. We have also used the representation (1.19) for the Wronskians.

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At this stage, we make use of the fact that $\Phi_n^{(i)}(x^{(i)})$ and $\Psi_n^{(i)}(x^{(i)})$ for $i = 1, 2$ are related by a constant of proportionality k_n, which, as (1.32) reveals, is *solely dependent on the impulse response data*, and hence identical for both beams. As a result (2.29) becomes

$$\begin{aligned}\mathscr{I}(\xi) = \sum_{n=1}^{\infty} \left[\omega_n^2 \Big/ \prod_{m\neq n} \left(1-\frac{\omega_n^2}{\omega_m^2}\right)\right] \Big[\!\Big[[r^{(1)}(L^{(1)})]^{-2} \\ \times \left\{\int_0^{\xi} \rho^{(2)}(x^{(2)}) \, [k_n \phi^{(2)}(x^{(2)}, \omega_n^2) - \psi^{(2)}(x^{(2)}, \omega_n^2)] \, h \, \frac{dx^{(2)}}{d\zeta} \, d\zeta \right\} \Psi_n^{(1)}(x^{(1)}) \\ + [r^{(2)}(L^{(2)})]^{-2} \left\{\int_{\xi}^{\Xi} \rho^{(2)}(x^{(2)}) \, \Psi_n^{(2)}(x^{(2)}) \, h \, \frac{dx^{(2)}}{d\zeta} \, d\zeta \right\} \\ \times [k_n \phi^{(1)}(x^{(1)}, \omega_n^2) - \psi^{(1)}(x^{(1)}, \omega_n^2)] \Big]\!\Big].\end{aligned}$$

This expression can be further simplified by means of (1.34′), i.e. since

$$k_n \phi^{(i)}(x^{(i)}, \omega_n^2) - \psi^{(i)}(x^{(i)}, \omega_n^2) = k_n^{(i)*} \, \Psi_n^{(i)}(x^{(i)}),$$

where, as (1.36) indicates,

$$[r^{(1)}(L^{(1)})]^2 k_n^{(1)*} = [r^{(2)}(L^{(2)})]^2 k_n^{(2)*} = k_n \Big/ F_1 \prod_m \left(1-\frac{\omega_n^2}{\nu_m^2}\right).$$

As a result

$$\begin{aligned}\mathscr{I}(\xi) = \sum_{n=1}^{\infty} \frac{1}{[r^{(1)}(L^{(1)}) \, r^{(2)}(L^{(2)})]^2} \, \frac{\omega_n^2 k_n}{F_1} \\ \times \left[\int_0^{\Xi} \rho^{(2)}(x^{(2)}) \, \Psi_n^{(2)}(x^{(2)}) \, h \, \frac{dx^{(2)}}{d\zeta} \, d\zeta \Big/ \prod_{m\neq n} \left(1-\frac{\omega_n^2}{\omega_m^2}\right) \prod_m \left(1-\frac{\omega_n^2}{\nu_m^2}\right)\right] \Psi_n^{(1)}(x^{(1)}). \qquad (2.30)\end{aligned}$$

To avoid problems with pathological beams, it is preferable to free (2.30) of the terms $r^{(1)}(L^{(1)})$, $r^{(2)}(L^{(2)})$. This can be done by defining the norm of $\Psi_n^{(i)}$ as

$$\|\Psi_n^{(i)}\|^2 = \int_0^{L^{(i)}} \rho^{(i)}(x^{(i)}) \, [\Psi_n^{(i)}(x^{(i)})]^2 \, dx^{(i)}. \qquad (2.31)$$

The superscripts are not necessary for the actual evaluation of the above norms; hence they shall be omitted temporarily. As given earlier the equations for $\Phi(x, -s)$ and $\Psi_n(x)$ are

$$\frac{d^2}{dx^2}\left(r \, \frac{d^2\Phi}{dx^2}\right) = s\rho\Phi,$$

and

$$\frac{d^2}{dx^2}\left(r \, \frac{d^2\Psi_n}{dx^2}\right) = -\omega_n^2 \rho \Psi_n.$$

Multiplying the first by Ψ_n, the second by Φ, subtracting, and integrating the resulting expression over $(0, L)$, we get after making use of (1.22) and (1.23)

$$\Psi_n(0) \, r(0) \, \Phi'''(0, s) = \omega_n^2 \left(1+\frac{s}{\omega_n^2}\right) \int_0^L \rho\Phi\Psi_n \, dx. \qquad (2.32)$$

The product representations (1.28) and (1.30*b*) enable us to write

$$\frac{r^4(L) \, F_1}{\omega_n^2} \prod_{m\neq n} \left(1+\frac{s}{\omega_m^2}\right) \prod_m \left(1-\frac{\omega_n^2}{\nu_m^2}\right) = \int_0^L \rho\Phi\Psi_n \, dx.$$

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It only remains now to let $s \to -\omega_n^2$ to see that

$$\frac{r^4(L)\,F_1}{\omega_n^2} \prod_{m \neq n} \left(1 - \frac{\omega_n^2}{\omega_m^2}\right) \prod \left(1 - \frac{\omega_n^2}{\nu_m^2}\right) = k_n \|\Phi_n\|^2 \tag{2.33}$$

and consequently (2.30) becomes

$$\mathscr{I}(\xi) = \sum_{n=1}^{\infty} \left\{ \int_0^{\Xi} \rho^{(2)}(x^{(2)})\, \Psi_n^{(2)}(x^{(2)})\, h \frac{\mathrm{d}x^{(2)}}{\mathrm{d}\zeta}\, \mathrm{d}\zeta \Big/ \|\Psi_n^{(1)}\| \, \|\Psi_n^{(2)}\| \right\} \Psi_n^{(1)}(x^{(1)}). \tag{2.34}$$

If the two beams were identical, then we could drop the superscripts altogether and get the classical eigenfunction expression

$$\mathscr{I}[\xi(x)] = \sum_{n=1}^{\infty} \left\{ \int_0^L \rho \Psi_n h[\zeta(x)]\, \mathrm{d}x \Big/ \|\Psi_n\|^2 \right\} \Psi_n(x)$$

as a special case of that more general eigenfunction expansion. Incidentally, here $\mathscr{I}(\xi)$ is just $h(\xi)$.

2.7. *Evaluation of $\mathscr{I}(\xi)$ via residue of pole at infinity*

An alternative expression for $\mathscr{I}(\xi)$ is obtained by taking the contour in the s-plane to be a very large circle and replacing $\Omega^{(2,1)}(x^{(2)}, x^{(1)}; -s)$ by its asymptotic expression for large $|s|$. The calculation is straightforward but tedious. We then perform the ξ integration over $(0, \xi)$ and (ξ, Ξ) asymptotically. This requires a mere integration by parts, provided, of course, that r and ρ are differentiable. The remaining integral over s is now trivial since to leading order the s-dependence of the integrand is s^{-1}. The result of these calculations is

$$\mathscr{I}(\xi) = \left\{ \frac{r^{(2)}[x^{(2)}(\xi)]}{r^{(1)}[x^{(1)}(\xi)]} \right\}^{\frac{1}{8}} \left\{ \frac{\rho^{(2)}[x^{(2)}(\xi)]}{\rho^{(1)}[x^{(1)}(\xi)]} \right\}^{\frac{3}{8}} h(\xi). \tag{2.35}$$

Equating the two expressions for $\mathscr{I}(\xi)$ we get

$$\left[\frac{r^{(2)}(x^{(2)})}{r^{(1)}(x^{(1)})}\right]^{\frac{1}{8}} \left[\frac{\rho^{(2)}(x^{(2)})}{\rho^{(1)}(x^{(1)})}\right]^{\frac{3}{8}} h \doteq \sum_n \left[\int_0^{\Xi} \rho^{(2)} \Psi_n^{(2)} h \frac{\mathrm{d}x^{(2)}}{\mathrm{d}\zeta}\, \mathrm{d}\zeta \Big/ \|\Psi_n^{(1)}\| \, \|\Psi_n^{(2)}\| \right] \Psi_n^{(1)}(x^{(1)}). \tag{2.36}$$

The left hand side can also be written in terms of a standard eigenfunction expansion:

$$\left\{\frac{r^{(2)}(x^{(2)})}{r^{(1)}(x^{(1)})}\right\}^{\frac{1}{8}} \left\{\frac{\rho^{(2)}(x^{(2)})}{\rho^{(1)}(x^{(1)})}\right\}^{\frac{3}{8}} h = \sum_n \left\{ \int_0^{\Xi} \rho^{(1)} \Psi_n^{(1)} \left[\frac{r^{(2)}}{r^{(1)}}\right]^{\frac{1}{8}} \left[\frac{\rho^{(2)}}{\rho^{(1)}}\right]^{\frac{3}{8}} h \frac{\mathrm{d}x^{(1)}}{\mathrm{d}\zeta}\, \mathrm{d}\zeta \Big/ \|\Psi_n^{(1)}\|^2 \right\} \Psi_n^{(1)}(x^{(1)}) \tag{2.37}$$

from which it follows that

$$\|\Psi_n^{(2)}\|^{-1} \int_0^{\Xi} \rho^{(2)} \Psi_n^{(2)} h \frac{\mathrm{d}x^{(2)}}{\mathrm{d}\zeta}\, \mathrm{d}\zeta = \|\Psi_n^{(1)}\|^{-1} \int_0^{\Xi} \rho^{(1)} \Psi_n^{(1)} \left[\frac{r^{(2)}}{r^{(1)}}\right]^{\frac{1}{8}} \left[\frac{\rho^{(2)}}{\rho^{(1)}}\right]^{\frac{3}{8}} h \frac{\mathrm{d}x^{(1)}}{\mathrm{d}\zeta}\, \mathrm{d}\zeta, \quad n = 1, 2, \ldots, \tag{2.38}$$

and since the function h is completely arbitrary

$$\{\rho^{(2)}[x^{(2)}(\xi)]\}^{\frac{3}{8}} \{r^{(2)}[x^{(2)}(\xi)]\}^{\frac{1}{8}} \Psi_n^{(2)}[x^{(2)}(\xi)] / \|\Psi_n^{(2)}\|$$
$$= \{\rho^{(1)}[x^{(1)}(\xi)]\}^{\frac{3}{8}} \{r^{(1)}[x^{(1)}(\xi)]\}^{\frac{1}{8}} \Psi_n^{(1)}[x^{(1)}(\xi)] / \|\Psi_n^{(1)}\|, \quad n = 1, 2, \ldots . \tag{2.39}$$

We must again digress before we can draw any conclusion from (2.39).

2.8. *The Liouville transformation and the canonical fourth-order equation*

We have been pushed first into the introduction of the variable

$$\xi = \int_0^x \left(\frac{\rho}{r}\right)^{\frac{1}{4}} \mathrm{d}x$$

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and now into considering the functions

$$U_n(\xi) = [\rho(x(\xi))]^{\frac{3}{8}}[r(x(\xi))]^{\frac{1}{8}}\Psi_n(x(\xi))/\|\Psi_n\|. \tag{2.40}$$

We can think of these new variables as constituting a Liouville transformation. In terms of them, the original eigenvalue problem,

$$\left.\begin{aligned} \frac{d^2}{dx^2}\left(r\frac{d^2\Psi_n}{dx^2}\right) &= \omega_n^2\rho\Psi_n, \\ \Psi_n'' = (r\Psi_n'')' &= 0 \quad \text{at} \quad x = 0, \\ \Psi_n = \Psi_n' &= 0 \quad \text{at} \quad x = L, \end{aligned}\right\}$$

is transformed into

$$\frac{d^4U_n}{d\xi^4} + \frac{d}{d\xi}\left[A(\xi)\frac{dU_n}{d\xi}\right] + B(\xi)U_n = \omega_n^2 U_n \tag{2.41}$$

with

$$\left.\begin{aligned} \frac{d^2U_n}{d\xi^2} + a\frac{dU_n}{d\xi} + bU_n &= 0 \\ \frac{d^3U_n}{d\xi^3} + c\frac{d^2U_n}{d\xi^2} + d\frac{dU_n}{d\xi} + eU_n &= 0 \end{aligned}\right\} \quad \text{at} \quad \xi = 0, \tag{2.42a}$$

and

$$U_n = dU_n/d\xi = 0 \quad \text{at} \quad \xi = \Xi. \tag{2.42b}$$

The coefficients $A(\xi)$ and $B(\xi)$ are related to ρ and r thus:

$$A = 4\frac{q_{\xi\xi}}{q} - 6\frac{q_\xi^2}{q^2} - 2\frac{q_\xi p_\xi}{qp} + \frac{p_{\xi\xi}}{p} - 2\frac{p_\xi^2}{p^2} \tag{2.43a}$$

$$\begin{aligned} B = {} & \frac{q_{\xi\xi\xi\xi}}{q} - 4\frac{q_\xi q_{\xi\xi\xi}}{q^2} - 2\frac{q_{\xi\xi}^2}{q^2} + 6\frac{q_\xi^2 q_{\xi\xi}}{q^3} + \frac{q_\xi p_{\xi\xi\xi}}{qp} + \frac{q_{\xi\xi}p_{\xi\xi}}{qp} \\ & - 3\frac{q_\xi^2 p_{\xi\xi}}{q^2 p} - \frac{q_\xi p_\xi p_{\xi\xi}}{qp^2} - 4\frac{q_\xi q_{\xi\xi} p_\xi}{q^2 p} + 6\frac{q_\xi^3 p_\xi}{q^3 p} - 2\frac{q_\xi^2 p_\xi^2}{q^2 p^2}, \end{aligned} \tag{2.43b}$$

where

$$p(\xi) = [\rho(x)/r(x)]^{\frac{1}{4}} \tag{2.44}$$

and

$$q(\xi) = \rho^{-\frac{3}{8}}(x)\,r^{-\frac{1}{8}}(x). \tag{2.45}$$

Also

$$\left.\begin{aligned} a &= 2\frac{q_\xi}{q} + \frac{p_\xi}{p}, \\ b &= \frac{q_{\xi\xi}}{q} + \frac{q_\xi p_\xi}{qp}, \\ c &= \frac{q_\xi}{q}, \\ d &= 3\frac{q_{\xi\xi}}{q} - 4\frac{q_\xi^2}{q^2} - 2\frac{q_\xi p_\xi}{qp} + \frac{p_{\xi\xi}}{p} - 2\frac{p_\xi^2}{p^2}, \\ e &= \frac{q_{\xi\xi\xi}}{q} - \frac{q_\xi q_{\xi\xi}}{q^2} + \frac{p_{\xi\xi}q_\xi}{pq} - 2\frac{p_\xi q_\xi^2}{pq^2} - 2\frac{p_\xi^2 q_\xi}{p^2 q}. \end{aligned}\right\} \tag{2.46}$$

Since we have been led to a consideration of the canonical fourth-order equation (2.41), one may ask why did we not consider this equation from the beginning? In fact, I have considered the inverse eigenvalue problem for this equation in a previous paper (Barcilon 1974).

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That analysis was valid only when the eigenvalues were simple. This assumption is true provided that the canonical equation can be written as

$$\alpha_0 \frac{d}{dx} \alpha_1 \frac{d}{dx} \alpha_2 \frac{d}{dx} \alpha_3 \frac{d}{dx} \alpha_4 U = \omega^2 U, \tag{2.47}$$

where $\alpha_0, \ldots, \alpha_4$ are positive, and if the boundary conditions satisfy certain requirements (Karlin 1971). Thus, one is not dealing with the most general canonical fourth-order equation, but with the subclass that has simple eigenvalues. Under those circumstances, it is preferable to deal with the beam equation.

Before resuming our discussion about uniqueness, we shall introduce some new variables which will simplify the formulas for $A(\xi)$, $B(\xi)$, a, …, e:

$$P(\xi) = p_\xi/p, \quad Q(\xi) = q_\xi/q, \tag{2.48}$$

or equivalently

$$p(\xi) = p(0) \exp\left[\int_0^\xi P(\zeta)\,d\zeta\right], \quad q(\xi) = q(0) \exp\left[\int_0^\xi Q(\zeta)\,d\zeta\right]. \tag{2.48'}$$

With these variables, (2.43) becomes

$$4Q_\xi + P_\xi - 2Q^2 - 2QP - P^2 - A = 0 \tag{2.49a}$$

and

$$Q_{\xi\xi\xi} + QP_{\xi\xi} + Q_\xi P_\xi - 4Q^2Q_\xi + Q_\xi^2 - 4QQ_\xi P + Q_\xi P^2 + 2QPP_\xi - 2Q^2P_\xi + Q^4 + 2Q^3P - 4Q^2P^2 - B = 0. \tag{2.49b}$$

or

$$\boldsymbol{Q}_\xi = \boldsymbol{F}(\boldsymbol{Q}) + \boldsymbol{A}, \tag{2.50}$$

where

$$\boldsymbol{Q} = \begin{bmatrix} Q \\ P \\ Q_\xi \\ P_\xi \\ Q_{\xi\xi} \end{bmatrix}, \quad \boldsymbol{F} = \begin{bmatrix} Q_\xi \\ P_\xi \\ Q_{\xi\xi} \\ -4Q_{\xi\xi} + 4QQ_\xi + 2Q_\xi P + 2QP_\xi + 2PP_\xi \\ +4QQ_{\xi\xi} - Q_\xi P_\xi - Q_\xi^2 + 2QQ_\xi P - 4QPP_\xi \\ -Q_\xi P + Q^4 + 2Q^3P - 4Q^2P^2, \end{bmatrix} \tag{2.51), (2.52}$$

and

$$\boldsymbol{A} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ A_\xi \\ B - QA_\xi \end{bmatrix}. \tag{2.53}$$

Finally, the equations (2.46) can be written as follows:

$$\boldsymbol{Q}(0) = \begin{bmatrix} c \\ a - 2c \\ b + c^2 - ac \\ d - 3b - 2c^2 + ac \\ e + cb - ac^2 - cd + 2c^3 \end{bmatrix}. \tag{2.54}$$

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2.9. *Uniqueness results*

To summarize, given two beams $\rho^{(1)}(x^{(1)})$, $r^{(1)}(x^{(1)})$ and $\rho^{(2)}(x^{(2)})$, $r^{(2)}(x^{(2)})$ with the same impulse response, provided that these functions are sufficiently smooth, we can make a Liouville transformation and write

$$\frac{d^4U_n}{d\xi^4}+\frac{d}{d\xi}\left(A^{(1)}\frac{dU_n}{d\xi}\right)+B^{(1)}U_n = \omega_n^2 U_n \tag{2.55a}$$

and
$$\frac{d^4U_n}{d\xi^4}+\frac{d}{d\xi}\left(A^{(2)}\frac{dU_n}{d\xi}\right)+B^{(2)}U_n = \omega_n^2 U_n. \tag{2.55b}$$

The fact that U_n has no superscript is a consequence of (2.39), namely

$$\frac{\Psi_n^{(1)}(x^{(1)})}{q^{(1)}(\xi)\,\|\Psi_n^{(1)}\|} = U_n(\xi) = \frac{\Psi_n^{(2)}(x^{(2)})}{q^{(2)}(\xi)\,\|\Psi_n^{(2)}\|} \quad \text{for all } n.$$

The coefficients $A^{(1)}$, $B^{(1)}$ and $A^{(2)}$, $B^{(2)}$ are obtained by substituting $p^{(1)}$, $q^{(1)}$ and $p^{(2)}$, $q^{(2)}$ in (2.43). Now, if we were to subtract (2.55b) from (2.55a) we would get

$$\frac{d}{d\xi}\left[(A^{(1)}-A^{(2)})\frac{dU_n}{d\xi}\right]+(B^{(1)}-B^{(2)})\,U_n = 0. \tag{2.56}$$

Since U_n satisfies (2.42b), we are forced to conclude that

$$\left.\begin{aligned} A^{(1)}(\xi) &= A^{(2)}(\xi) = A(\xi),\\ B^{(1)}(\xi) &= B^{(2)}(\xi) = B(\xi),\end{aligned}\right\} \tag{2.57}$$

or else U_n would be identically zero. Similarly, a consideration of the boundary conditions (2.42a) at $\xi = 0$, implies that

$$\left.\begin{aligned} a^{(1)} &= a^{(2)} = a,\\ b^{(1)} &= b^{(2)} = b,\\ c^{(1)} &= c^{(2)} = c,\\ d^{(1)} &= d^{(2)} = d,\\ e^{(1)} &= e^{(2)} = e.\end{aligned}\right\} \tag{2.58}$$

(This can be seen, for instance, by considering eigenfunctions associated with large values of n.) As a result, the vectors $\boldsymbol{Q}^{(1)}$ and $\boldsymbol{Q}^{(2)}$ are solutions of the same differential equation (2.50) with initial conditions (2.54). If the solution of this initial value problem exists, then it is unique since $\boldsymbol{F}$ is Lipschitz. Consequently

$$\boldsymbol{Q}^{(1)}(\xi) = \boldsymbol{Q}^{(2)}(\xi)$$

and in particular
$$Q^{(1)}(\xi) = Q^{(2)}(\xi), \quad P^{(1)}(\xi) = P^{(2)}(\xi). \tag{2.59}$$

This, in turn, implies that
$$p^{(1)}(\xi)/p^{(1)}(0) = p^{(2)}(\xi)/p^{(2)}(0)$$

and
$$q^{(1)}(\xi)/q^{(1)}(0) = q^{(2)}(\xi)/q^{(2)}(0),$$

and finally in view of (2.21) and (2.22),

$$\rho^{(1)}(x^{(1)}(\xi)) = \rho^{(2)}(x^{(2)}(\xi)), \quad r^{(1)}(x^{(1)}(\xi)) = r^{(2)}(x^{(2)}(\xi)). \tag{2.60}$$

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For the smooth class of functions that we have been considering, we can in fact go further and see that $L^{(1)} = L^{(2)}$, and hence

$$\rho^{(1)}(x) = \rho^{(2)}(x), \quad r^{(1)}(x) = r^{(2)}(x). \tag{2.61}$$

However, because of the pathological beams previously discussed, I suspect that (2.60) is the best result possible. Thus, if we are willing to disregard the pathological beams, we can say that the impulse response determines a beam uniquely.

3. Existence and construction of solution

In this section we shall outline a procedure for reconstructing a beam from its impulse response. This procedure was previously presented by Barcilon (1979*a*, *b*). The central idea revolves around the use of continued fractions and owes much to the work of Krein (1951, 1952*a*).

The procedure for reconstructing $\rho(x)$ and $r(x)$, given F_1, F_2 and $\{\omega_n, \nu_n, \mu_n\}_1^\infty$, will work provided that the data are bona fide data. Thus, we shall need elaborate criteria for recognizing whether or not a solution exists for a given impulse response. We do not have definitive results about all such criteria: we shall derive a few and suggest the need for more.

Before we embark on this programme, it might be useful to review the inverse problem for a vibrating string. The impulse response for a string in the free–fixed configuration consists of (i) the length L of the string and (ii) the spectra $\{\lambda_n\}_1^\infty$ and $\{\mu_n\}_1^\infty$ associated with the fixed–fixed and free–fixed configurations. The numbers L, $\{\mu_n, \lambda_n\}_1^\infty$ constitute a bona fide impulse response if they satisfy the following conditions:

(i) asymptotic behaviour,

$$\lambda_n \sim \mu_n \sim n\pi \Big/ \int_0^L \rho^{\frac{1}{2}}(x)\,\mathrm{d}x, \quad \text{as} \quad n \to \infty; \tag{3.1}$$

(ii) interlacing,

$$\mu_1 < \lambda_1 < \mu_2 < \lambda_2 < \ldots; \tag{3.2}$$

(iii) global condition,

$$\sum_{n=1}^{\infty} \left[\mu_n^2 \prod_{k\neq n}^{\infty} \left(1 - \frac{\mu_n^2}{\mu_k^2}\right) \prod_{k}^{\infty} \left(1 - \frac{\mu_n^2}{\lambda_k^2}\right)\right]^{-1} < \infty. \tag{3.3}$$

The asymptotic condition is easy to understand and can be deduced straightforwardly from the direct problem. The interlacing condition is also sensible: the μ are associated with a configuration freer than that for the λ. Less obvious is the fact that this condition ensures the positivity of $\rho(x)$. This is connected to a theorem of Stieltjes (1894) which we shall discuss in §3.7. The third condition ensures that the total mass of the string is finite. It plays an essential role in the construction of the solution to the inverse problem for a vibrating string. We would like to find the conditions for the beam analogous to (3.1)–(3.3) for the string.

3.1. *Interlacing of eigenvalues*

One would expect that the following ordering would hold:

$$\omega_1 < \sigma_1 < \nu_1 < \mu_1 < \lambda_1 < \ldots < \omega_i < \sigma_i < \nu_i < \mu_i < \lambda_i < \ldots.$$

However, this is *not* the case. Indeed, by considering the homogeneous beam (i.e. ρ and r constants), we can see that the spectrum associated with the clamped–clamped configuration,

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namely $\{\lambda_n\}_1^\infty$, does not interlace with that for the free–clamped one, i.e. $\{\omega_n\}_1^\infty$ (Platzman private communication).

To elucidate this question, we appeal to a paper of Krein (1939). If we were to apply the theorem stated in that paper to the beam considered here, we would get

$$\omega_i < \lambda_i < \omega_{i+2}, \tag{3.4a}$$

$$\sigma_i < \lambda_i < \sigma_{i+2}, \tag{3.4b}$$

$$\nu_i < \lambda_i < \nu_{i+2}, \tag{3.4c}$$

$$\mu_i < \lambda_i < \mu_{i+2}. \tag{3.4d}$$

These results are general results stemming from the properties of Green functions of fourth-order operators that are, so-called, 'oscillating'. These results can be improved for the particular case of the beam operator. The first step consists of examining the first eigenvalues of our five eigenvalue problems. We start by considering μ_1 and ν_1. To that effect we introduce a hybrid eigenvalue problem as follows:

$$\left.\begin{aligned} (ry'')'' &= \Lambda^2 \rho y, \quad 0 < x < L, \\ \alpha y - (1-\alpha) y' &= y'' = 0, \quad x = 0, \\ y &= y' = 0, \quad x = L, \end{aligned}\right\} \tag{3.5}$$

where α is a parameter ranging over (0, 1). Clearly

$$\Lambda_n(0) = \nu_n, \tag{3.6a}$$

and

$$\Lambda_n(1) = \mu_n. \tag{3.6b}$$

The adjoint problem, which will also enter into our discussion, is

$$\left.\begin{aligned} (r\eta'')'' &= \Lambda^2 \rho\eta, \quad 0 < x < L, \\ \eta &= \alpha(r\eta'') - (1-\alpha)(r\eta'')' = 0, \quad x = 0, \\ \eta &= \eta' = 0, \quad x = L. \end{aligned}\right\} \tag{3.7}$$

After differentiation with respect to the parameter α, we can easily see that

$$\frac{d\Lambda^2}{d\alpha} = -y'(0,\alpha)(r\eta'')(0,\alpha) \Big/ \alpha(1-\alpha)\int_0^L \rho y\eta \, dx. \tag{3.8}$$

We shall specialize this result to the first eigensolutions of (3.5) and (3.7) and appeal to a result of Gantmakher† (1936), namely that

$$y_1(x,\alpha) > 0, \quad \eta_1(x,\alpha) > 0 \quad \text{for} \quad 0 < x < L. \tag{3.9}$$

Consequently

$$y_1(0,\alpha) \geqslant 0$$

and in view of the boundary condition at $x = 0$

$$y_1'(0,\alpha) \geqslant 0. \tag{3.10}$$

We prove next that

$$(r\eta_1'')(0,\alpha) \leqslant 0. \tag{3.11}$$

Let us assume the contrary. Then by continuity there exists an interval, say $(0, \xi)$, over which

$$r\eta_1'' > 0.$$

† Strictly, we should prove first that (3.5) and (3.7) can be transformed into integral equations with oscillating kernels. This is indeed so thanks to a theorem of Karlin (1971).

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This interval must be smaller than the length of the beam. Indeed, otherwise η_1' would be an increasing function throughout $(0, L)$ and since $\eta_1'(0, \alpha)$ must be positive, it would follow that $\eta_1'(L, \alpha)$ is also positive. But this is impossible on account of the boundary condition. Thus $0 < \xi < L$. We can be more specific and define ξ such that

$$(r\eta_1'')(\xi, \alpha) = 0.$$

Now, integrating (3.7) twice from 0 to ξ, we see that

$$-[(r\eta_1'')(0, \alpha) + \xi(r\eta_1'')'(0, \alpha)] = \Lambda_1^2 \int_0^\xi (\xi - t)\rho\eta_1 \mathrm{d}t,$$

and since $(r\eta_1'')(0, \alpha)$ is positive, it follows from the boundary condition that $(r\eta_1'')'(0, \alpha)$ is also positive. Hence, the left side of the above equation is negative whereas the right side is positive. Therefore (3.11) must be true and as a result, the derivative of Λ_1^2 with respect to α is positive, i.e.

$$\nu_1 < \mu_1. \tag{3.12}$$

By considering a problem similar to (3.5) but with the boundary conditions

$$\alpha y - (1-\alpha) y' = (ry'')' = 0 \quad \text{at} \quad x = 0$$

we could show that
$$\sigma_1 < \nu_1. \tag{3.13}$$

Similarly, we could also prove that
$$\omega_1 < \sigma_1. \tag{3.14}$$

Combining all of these results, we get

$$\omega_1 < \sigma_1 < \nu_1 < \mu_1 < \lambda_1. \tag{3.15}$$

The technique used to establish (3.15) is not suitable for the higher eigenvalues. To order them I have found that the introduction of certain auxiliary functions is very helpful.

3.2. *Auxiliary functions*

Recall that $u(x, -s)$ and $v(x, -s)$ were fundamental solutions of the beam equation chosen in such a way as to satisfy the clamped boundary conditions at $x = L$ (see (1.7)). Recall also the forced problem (2.3)–(2.4) which led to the impulses response when $\rho(x)\eta(x)$ was set equal to $\delta(x)$. We can rewrite this problem as follows:

$$\left.\begin{aligned} (ry'')'' &= -s\rho y, \\ ry'' = (ry'')' - 1 &= 0, \quad x = 0, \\ y = y' &= 0, \quad x = L. \end{aligned}\right\} \tag{3.16}$$

We can avail ourselves of the functions u and v to solve this problem. In fact

$$y(x, -s) = -\frac{u(x, -s)(rv'')(0, -s) - v(x, -s)(ru'')(0, -s)}{(ru'')(0, -s)(rv'')'(0, -s) - (rv')(0, -s)(ru'')'(0, -s)}, \tag{3.17}$$

and as a result, the impulse response given in (2.9) can also be written as

$$y(0, -s) = -\frac{u(0, -s)(rv'')(0, -s) - v(0, -s)(ru'')(0, -s)}{(ru'')(0, -s)(rv'')'(0, -s) - (rv'')(0, -s)(ru'')'(0, -s)} \tag{3.18a}$$

and
$$\theta(0, -s) = -\frac{u'(0, -s)(rv'')(0, -s) - v'(0, -s)(ru'')(0, -s)}{(ru'')(0, -s)(rv'')'(0, -s) - (rv'')(0, -s)(ru'')'(0, -s)}. \tag{3.18b}$$

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These expressions suggest that we define the following functions:

$$Y(x,\omega^2) = u(x,\omega^2)\,(rv'')\,(x,\omega^2)-v(x,\omega^2)\,(ru'')\,(x,\omega^2), \tag{3.19}$$

$$\Theta(x,\omega^2) = u'(x,\omega^2)\,(rv'')\,(x,\omega^2)-v'(x,\omega^2)\,(ru'')\,(x,\omega^2), \tag{3.20}$$

$$D(x,\omega^2) = (ru'')\,(x,\omega^2)\,(rv'')'\,(x,\omega^2)-(rv'')\,(x,\omega^2)\,(ru'')'\,(x,\omega^2), \tag{3.21}$$

and that we work *directly* with them rather than with u and v. Numerically, this approach has the advantage of minimizing the loss of accuracy due to cancellations in the computation of Y, Θ and D.

We can look upon Y, Θ and D as determinants obtained by considering the 2×2 sub-matrices arising from the 4×2 matrix

$$\begin{bmatrix} u & v \\ u' & v' \\ ru'' & rv'' \\ (ru'')' & (rv'')' \end{bmatrix}.$$

Three other such determinants can be formed, namely

$$I(x,\omega^2) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}, \tag{3.22}$$

$$J(x,\omega^2) = \begin{vmatrix} u & v \\ (ru'')' & (rv'')' \end{vmatrix}, \tag{3.23}$$

and

$$K(x,\omega^2) = \begin{vmatrix} u' & v' \\ (ru'')' & (rv'')' \end{vmatrix}. \tag{3.24}$$

It is simple to check that these six auxiliary functions satisfy the following differential equations:

$$I' = r^{-1}Y, \quad Y' = \Theta + J, \tag{3.25a, b}$$

$$J' = K, \quad \Theta' = K, \tag{3.25c, d}$$

$$K' = r^{-1}D-\omega^2\rho I, \quad D' = -\omega^2\rho Y, \tag{3.25e, f}$$

where primes indicate differentiation with respect to x. These auxiliary equations, rather than the beam equation, will occupy centre stage during the reconstruction procedure. Note that in view of (1.7), the boundary conditions associated with (3.25) are

$$I = Y = J = \Theta = K = D-r^2(L) = 0 \quad \text{at} \quad x = L. \tag{3.26}$$

Keeping in mind the definitions (3.19)–(3.24) of the auxiliary functions, it is simple to show that

$$I(0,\omega^2) = r^2(L)\,(F_0F_2-F_1^2)\prod_{n=1}^{\infty}\left(1-\frac{\omega^2}{\lambda_n^2}\right), \tag{3.27a}$$

$$Y(0,\omega^2) = -r^2(L)\,F_2\prod_{n=1}^{\infty}\left(1-\frac{\omega^2}{\mu_n^2}\right), \tag{3.27b}$$

$$J(0,\omega^2) = \Theta(0,\omega^2) = r^2(L)\,F_1\prod_{n=1}^{\infty}\left(1-\frac{\omega^2}{\nu_n^2}\right), \tag{3.27c}$$

$$K(0,\omega^2) = -r^2(L)\,F_0\prod_{n=1}^{\infty}\left(1-\frac{\omega^2}{\sigma_n^2}\right), \tag{3.27d}$$

$$D(0,\omega^2) = r^2(L)\prod_{n=1}^{\infty}\left(1-\frac{\omega^2}{\omega_n^2}\right). \tag{3.27e}$$

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In these expressions F_1 and F_2 are the first and second moments of the flaccidity defined in (1.27) and (1.29) while F_0 is the zero moment, namely

$$F_0 = \int_0^L \frac{\mathrm{d}t}{r(t)}. \tag{3.28}$$

Note that (3.25*c*) and (3.25*d*) together with the boundary conditions (3.26) imply that

$$\Theta(x, \omega^2) = J(x, \omega^2). \tag{3.29}$$

This result is easy to understand: it means that the eigenfrequencies of the truncated beam (x, L) are identical if the boundary conditions at the left end are $y = (ry'')' = 0$ or $y' = ry'' = 0$. As we already know, these two problems are mutually adjoint.

In deriving the expression (3.27*a*) for $I(0, \omega^2)$ we made use of an identity for the auxiliary functions that will play a crucial role, namely

$$-I(x, \omega^2)\,D(x, \omega^2) + Y(x, \omega^2)\,K(x, \omega^2) - \Theta^2(x, \omega^2) = 0. \tag{3.30}$$

This identity can be established from the differential equations (3.25). However, its character is algebraic and it is best obtained from the very definitions of the auxiliary functions.

The product representations (3.27) for $x = 0$, can be viewed as special cases of product representations for arbitrary x. Indeed, the auxiliary functions are entire functions of ω^2 of order $\frac{1}{4}$. Consequently, we can write

$$I(x, \omega^2) = r^2(L)\left[\int_x^L \frac{\mathrm{d}t}{r}\int_x^L \frac{(t-x)^2}{r}\,\mathrm{d}t - \left(\int_x^L \frac{t-x}{r}\,\mathrm{d}t\right)^2\right]\prod_{n=1}^{\infty}\left(1-\frac{\omega^2}{\lambda_n^2(x)}\right), \tag{3.27$'a$}$$

$$Y(x, \omega^2) = -r^2(L)\int_x^L \frac{(t-x)^2}{r}\,\mathrm{d}t\prod_{n=1}^{\infty}\left(1-\frac{\omega^2}{\mu_n^2(x)}\right), \tag{3.27$'b$}$$

$$\Theta(x, \omega^2) = r^2(L)\int_x^L \frac{t-x}{r}\,\mathrm{d}t\prod_{n=1}^{\infty}\left(1-\frac{\omega^2}{\nu_n^2(x)}\right), \tag{3.27$'c$}$$

$$K(x, \omega^2) = -r^2(L)\int_x^L \frac{\mathrm{d}t}{r}\prod_{n=1}^{\infty}\left(1-\frac{\omega^2}{\sigma_n^2(x)}\right), \tag{3.27$'d$}$$

$$D(x, \omega^2) = r^2(L)\prod_{n=1}^{\infty}\left(1-\frac{\omega^2}{\omega_n^2(x)}\right). \tag{3.27$'e$}$$

The functions $\lambda_n(x)$, $\mu_n(x)$, $\nu_n(x)$, $\sigma_n(x)$ and $\omega_n(x)$ are the nth eigenfrequencies of a truncated beam (x, L) clamped at the right end and satisfying respectively the clamped, supported, non-self-adjoint, Rayleigh and free boundary conditions at the left end. These eigenfrequencies are real, positive and simple as a consequence of the theory of oscillating kernels.

These eigenfrequencies have an additional property: they are increasing functions of x. This is easy to understand physically: as the section $(0, x)$ is stripped off the original beam, the inertia of the beam is decreased and as a result the natural frequencies are increased. We sketch a proof of this statement, or more specifically of the fact that $\mathrm{d}\lambda_n^2(x)/\mathrm{d}x > 0$. To that effect consider the truncated beam (α, L). Then

$$(ry_n'')'' = \lambda_n^2 \rho y_n, \quad \alpha < x < L,$$

$$y_n(\alpha) = y_n'(\alpha) = 0,$$

$$y_n(L) = y_n'(L) = 0.$$

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Indicating a derivative with respect to α by a dot, we deduce that

$$(r\dot{y}_n'')'' = 2\lambda_n\dot{\lambda}_n\rho y_n + \lambda_n^2\rho\dot{y}_n, \tag{3.31}$$

and from the boundary conditions

$$\dot{y}_n + y_n' = \dot{y}_n' + y_n'' = 0 \quad \text{at} \quad x = \alpha,$$
$$\dot{y}_n = \dot{y}_n' = 0 \quad \text{at} \quad x = L.$$

Multiplying (3.31) by y_n and integrating over (α, L) we get

$$-r(\alpha)\,y_n''(\alpha, \alpha)\,\dot{y}_n'(\alpha, \alpha) = 2\lambda_n\dot{\lambda}_n\,\|y_n\|^2,$$

which, after some simple manipulations, yields

$$\frac{\mathrm{d}\lambda_n(\alpha)}{\mathrm{d}\alpha} = \frac{r(\alpha)\,[y_n''(\alpha, \alpha)]^2}{2\lambda_n(\alpha)\,\|y_n\|^2}.$$

A similar procedure could be used for the other spectra.

3.3. *The grand interlacing*

We are now in a position to resume our discussion about the interlacing of the various spectra.

Let us consider the curve $I(x, \omega^2)$ for a fixed value of ω^2 lying between λ_1^2 and λ_2^2 (see figure 2). The curve $I(x, \omega^2)$ is positive for x in $(0, l)$ and negative for x in (l, L) where

$$\lambda_1^2(l) = \omega^2.$$

Equation (3.25*a*) implies that the zeros of I' coincide with the curves $\omega^2 = \mu_n^2(x)$. Therefore I' vanishes only once in (l, L) since an immediate generalization of (3.15) would yield

$$\mu_1(x) < \lambda_1(x).$$

Over $(0, l)$, I' could either vanish or not vanish. If I' were different from zero for all x in $(0, l(\omega^2))$ and for all ω in (λ_1, λ_2), then we would have

$$\mu_2 > \lambda_2$$

which contradicts (3.4*d*). On the other hand if I' vanished three or more times, then we would have

$$\mu_4 < \lambda_2,$$

which also contradicts (3.4*d*). Therefore I' can vanish either once or twice in $(0, l)$. Let us examine the second case more closely. Graphically, the situation is as indicated in figure 3. Consider the structure of I on the straight line $\omega = \lambda_2$. Certainly

$$I(0, \lambda_2^2) = 0, \tag{3.32}$$

and

$$I(x, \lambda_2^2) \leqslant 0 \tag{3.33}$$

over the interval $(0, l(\lambda_2^2))$. Now $\quad Y(0, \lambda_2^2) > 0.$

But, because of the auxiliary equation (3.25*a*), this inequality implies that

$$I'(0, \lambda_2^2) > 0$$

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and hence for small values of x, it follows from (3.32) that

$$I(x, \lambda_2^2) > 0$$

which contradicts (3.33). Hence I' can only vanish once and as a result

$$\mu_1 < \lambda_1 < \mu_2 < \lambda_2.$$

Repeating this procedure, we can conclude that

$$\mu_i < \lambda_i < \mu_{i+1} < \lambda_{i+1}. \tag{3.34}$$

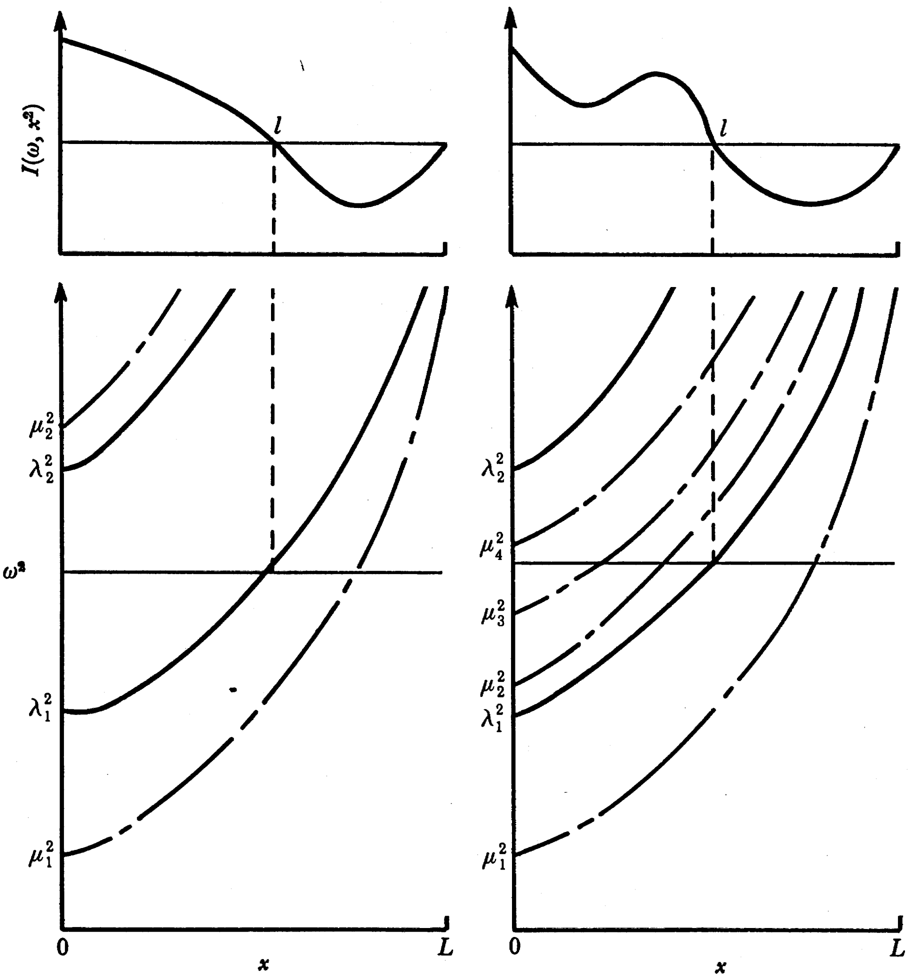

FIGURE 2. Sketches of the auxiliary function $I(x, \omega^2)$ for a value of ω^2 lying between λ_1^2 and λ_2^2. The number of local extrema depends on the relation between the λ- and μ-spectra.

By exploiting (3.25b), or rather $\quad Y' = 2\Theta,$

we could show similarly that $\quad \nu_i < \mu_i < \nu_{i+1} < \mu_{i+1}. \quad$ (3.35)

Similarly, (3.25d) and (3.25f) would yield

$$\sigma_i < \nu_i < \sigma_{i+1} < \nu_{i+1} \tag{3.36}$$

and $\quad \omega_i < \mu_i < \omega_{i+1} < \mu_{i+1}. \quad$ (3.37)

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Additional interlacings can be obtained by means of the quadratic identity (3.30). Indeed, setting $x = 0$ and $\omega = \omega_n$ in (3.30), we see that

$$Y(0, \omega_n^2) K(0, \omega_n^2) - \Theta^2(0, \omega_n^2) = 0,$$

which, together with (3.37) implies that

$$\omega_i < \sigma_i < \omega_{i+1} < \sigma_{i+1}. \tag{3.38}$$

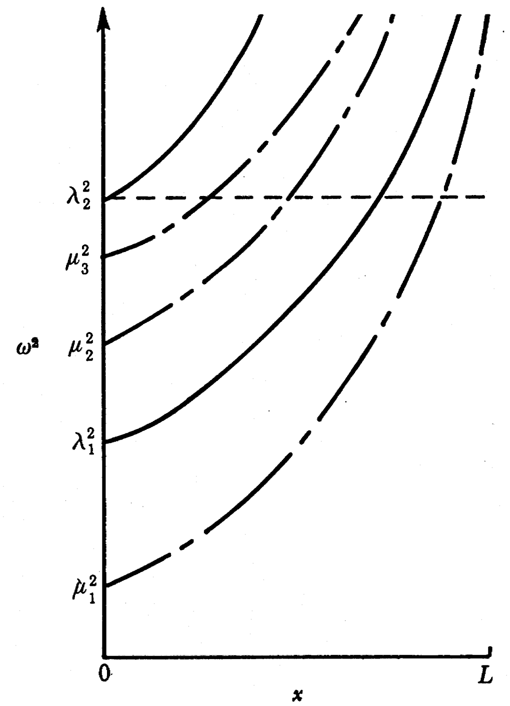

FIGURE 3. The ω^2,x-plane. The solid curves $\lambda_1^2(x)$ and $\lambda_2^2(x)$ represent the first and second eigenvalues of the truncated beam spanning the interval (x, L) in the clamped–clamped configuration. The dotted curves represent the corresponding eigenvalues for the supported–clamped configuration. By considering $I(x, \lambda_2^2)$ we can show that this arrangement of eigenvalues is not permissible.

Had we set $x = 0$ and $\omega = \sigma_n$ in (3.30), we would have obtained

$$-I(0, \sigma_n^2) D(0, \sigma_n^2) - \Theta^2(0, \sigma_n^2) = 0,$$

which, together with (3.38) implies that

$$\sigma_i < \lambda_i < \sigma_{i+1} < \lambda_{i+1}. \tag{3.39}$$

By combining all of these interlacings we can write the *grand* interlacing

$$\dots < \sigma_i < \nu_i < \mu_i < \begin{pmatrix} \lambda_i \\ \omega_{i+1} \end{pmatrix} < \sigma_{i+1} < \nu_{i+1} < \mu_{i+1} < \begin{pmatrix} \lambda_{i+1} \\ \omega_{i+1} \end{pmatrix} < \dots \tag{3.40}$$

The term $\binom{\lambda_i}{\omega_{i+1}}$ indicates that the order between these two eigenvalues cannot be decided. As we mentioned, the homogeneous beam provides an instance where one can check that the λ and the ω do not interlace.

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Clearly, the data ought to satisfy (3.40) for a solution to exist. More specifically, since only $\{\omega_n\}$, $\{\nu_n\}$ and $\{\mu_n\}$ are given, the data should satisfy the *small* interlacing, namely

$$\omega_1 < \nu_1 < \mu_1 < \omega_2 < \nu_2 < \mu_2 < \dots. \tag{3.41}$$

We are prepared, of course, to require that the ω, ν and μ spectra have the asymptotic behaviour given in (1.38) and perhaps some other gross condition(s) akin to (3.3) for the vibrating string. But, as we shall soon see, the situation is far more complicated and other conditions must come into play. Some of these additional conditions arise quite naturally in the process of constructing the solution to the inverse problem. Therefore, we must examine this process; the first step is a discretization of the direct problem.

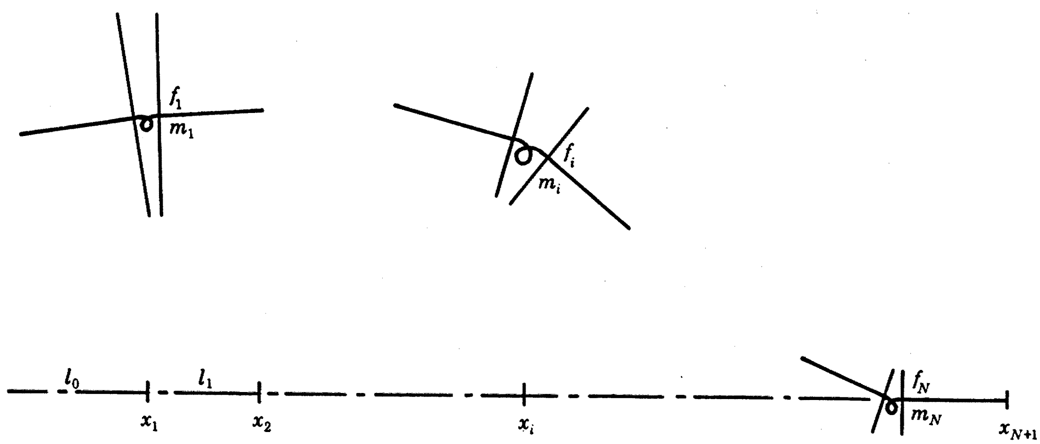

FIGURE 4. The discrete N-beam. Its structure consists of N clothes-pins located at $x_1, x_2, \dots, x_N$. These clothes-pins have masses $\{m_i\}$ and flaccidity $\{f_i\}$ and are connected by weightless, infinitely rigid rods of length $\{l_i\}$.

3.4. *A useful discretization*

The basic idea for solving the inverse problem is to look upon the beam with given spectra $\{\omega_n \nu_n \mu_n\}_1^\infty$ as the limit of a simpler beam with truncated spectra $\{\omega_n \nu_n \mu_n\}_1^{N-1}$ as the number of eigenfrequencies tends to infinity.

Beams with a finite number of eigenfrequencies must have a finite number of degrees of freedom. Thus, they must be made up by a finite number of point masses. Consequently, their density structure is of the form

$$\rho(x) = \sum_{i=1}^{N-1} m_i \, \delta(x - x_i). \tag{3.42}$$

The structure of the flexural rigidity is not dictated as clearly. One useful approximation is

$$\frac{1}{r(x)} = \sum_{i=1}^{N} f_i \, \delta(x - x_i). \tag{3.43}$$

I have discussed already such discrete N-beams (Barcilon 1979*a*, *b*): they can be thought of as being made of $N+1$ weightless segments of length $\{l_i\}_0^N$ and of infinite rigidity, connected by clothes-pin-like devices of mass $\{m_i\}_1^{N-1}$ and 'flaccidity' (or limpness) $\{f_i\}_1^N$ (see figure 4). These elastic joints are located at $\{x_i\}_1^N$ where

$$x_{i+1} - x_i = l_i, \quad i = 0, 1, \dots, N. \tag{3.44}$$

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It is important to realize that the structure (3.42) of the density is solely responsible for the finite number of eigenfrequencies of the resulting beam. The structure (3.43) of the flaccidity is chosen for convenience. In fact, other forms could have been chosen that might indeed be preferable in other problems. For instance, we could have written

$$1/r(x) = 1/r_{i-1} \quad \text{for} \quad x_{i-1} < x < x_i,$$

i.e. we could have considered the weightless segments to have finite constant rigidity.

Using steps similar to those in Barcilon (1979*b*) we can deduce that the discrete auxiliary variables satisfy the following difference equations:

$$I_i = I_{i-1} + f_i Y_i, \tag{3.45a}$$

$$Y_i = Y_{i-1} + l_{i-1}\Theta_{i-1} + l_{i-1} J_i, \tag{3.45b}$$

$$J_i = J_{i-1} + l_{i-1} K_{i-1}, \tag{3.45c}$$

$$\Theta_i = \Theta_{i-1} + l_{i-1} K_{i-1}, \tag{3.45d}$$

$$K_i = K_{i-1} + f_i D_i - \omega^2 m_i I_{i-1}, \tag{3.45e}$$

$$D_i = D_{i-1} - \omega^2 m_i Y_i. \tag{3.45f}$$

We can check that the discrete version of the quadratic identity (3.30), namely

$$-I_i D_i + Y_i K_i - \Theta_i^2 = 0 \tag{3.46}$$

is compatible with the equations (3.45). Equation (3.46) takes into account the fact that

$$J_i = \Theta_i, \tag{3.47}$$

which follows from (3.45*c*), (3.45*d*) and the end conditions

$$I_N = Y_N = J_N = \Theta_N = K_N = D_N - 1 = 0. \tag{3.48}$$

Even though the generic N-beam under discussion has a length x_{N+1}, we ended up applying the boundary conditions at x_N. In fact, the length of the last infinitely rigid, weightless segment l_N as well as the mass of the last clothes-pin m_N do not enter into the equations of the discrete beam. This is a consequence of the clamped boundary conditions at the right end. Indeed, the last segment remains stationary during the vibration. Similarly, the last clothes-pin flexes but does not move up or down. This situation is reminiscent of that for the pathological beams previously discussed.

Starting from the values of the discrete auxiliary variables at $x = x_N$ and making use of (3.45), we can show that $I_i(\omega^2)$ is a polynomial in ω^2 of degree $N-i-2$ whereas $Y_i(\omega^2)$, $\Theta_i(\omega^2)$, $K_i(\omega^2)$ and $D_i(\omega^2)$ are polynomials of degree $N-i-1$. In particular, at the point $x = 0$ associated with $i = 0$, we have

$$I_0(\omega^2) = \left[\sum_1^N f_i \sum_1^N x_i^2 f_i - \left(\sum_1^N x_i f_i\right)^2\right] \prod_{n=1}^{N-2}\left(1 - \frac{\omega^2}{\lambda_n^2}\right), \tag{3.49a}$$

$$Y_0(\omega^2) = -\left[\sum_1^N x_i^2 f_i\right] \prod_{n=1}^{N-1}\left(1 - \frac{\omega^2}{\mu_n^2}\right), \tag{3.49b}$$

$$\Theta_0(\omega^2) = \left[\sum_1^N x_i f_i\right] \prod_{n=1}^{N-1}\left(1 - \frac{\omega^2}{\nu_n^2}\right), \tag{3.49c}$$

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$$K_0(\omega^2) = -\left[\sum_1^N f_i\right] \prod_{n=1}^{N-1} \left(1-\frac{\omega^2}{\sigma_n^2}\right), \tag{3.49d}$$

$$D_0(\omega^2) = \prod_{n=1}^{N-1} \left(1-\frac{\omega^2}{\omega_n^2}\right). \tag{3.49e}$$

These expressions are the discrete analogues of (3.27). We are now in a position to outline a program for solving the inverse problem. Given the impulse response, i.e. given $\{\omega_n, \nu_n, \mu_n\}_1^\infty$, F_1 and F_2, we shall attempt to construct that N-beam whose eigenfrequencies are $\{\omega_n, \nu_n, \mu_n\}_1^{N-1}$ and such that

$$\sum_1^N x_i f_i = F_1, \quad \sum_1^N x_i^2 f_i = F_2. \tag{3.50}$$

In other words, given the $3N-1$ bits of data $\{\omega_n, \nu_n, \mu_n\}_1^{N-1}$, F_1, F_2 we set out to find the $3N-1$ unknowns $\{x_i\}_1^N$, $\{m_i\}_1^{N-1}$, $\{f_i\}_1^N$.

3.5. *Construction of $I_0(\omega^2)$ and $K_0(\omega^2)$*

With the given data, we can construct the polynomials $Y_0(\omega^2)$, $\Theta_0(\omega^2)$ and $D_0(\omega^2)$. By means of the polynomials and the quadratic identity (3.46) for $i = 0$, we can find the value of I_0 at $N-1$ points. Indeed, by setting $\omega = \mu_n$ in (3.46) we deduce that

$$I_0(\mu_n^2) = -\Theta_0^2(\mu_n^2)/D_0(\mu_n^2), \quad n = 1, 2, \ldots, N-1. \tag{3.51}$$

$I_0(\omega^2)$, which as indicated by (3.49) is a polynomial of degree $N-2$, is therefore completely determined:

$$I_0(\omega^2) = -\sum_{n=1}^{N-1} \frac{\Theta_0^2(\mu_n^2)}{D_0(\mu_n^2)} \left[\prod_{k\neq n}^{N-1} \left(1-\frac{\omega^2}{\mu_k^2}\right) \Big/ \prod_{k\neq n}^{N-1} \left(1-\frac{\mu_n^2}{\mu_k^2}\right)\right]. \tag{3.52}$$

Note that the zeros of the above polynomial are real and positive. Indeed, since ω_i and μ_i interlace, the terms in the sequence $\{D_0(\mu_n^2)\}_1^{N-1}$ alternate in sign. As a result the sign of $I_0(\mu_n^2)$ alternates with n. We shall denote the zeros of $I_0(\omega^2)$ by $\{\lambda_n^{(N)}\}_1^{N-2}$, the superscript N being used to remind us that these eigenvalues are associated with the N-beam. We shall drop this superscript whenever there is no possible source of confusion.

We have just outlined a procedure for finding the eigenfrequencies of the N-beam in the clamped–clamped configuration. The fact that we can find these eigenfrequencies is not surprising since F_1, F_2 and $\{\omega_n, \nu_n, \mu_n\}_1^{N-1}$ define the N-beam completely. What is surprising is that we can do it without having to first solve for the structure of the N-beam! In the same way, we can deduce $K_0(\omega^2)$ and its zeros. We should point out that the fact that we were successful in determining $I_0(\omega^2)$ and $K_0(\omega^2)$ is related to the way in which the given spectra entered into the quadratic identity. As discussed in Barcilon (1979*b*) three spectra and two gross constants are sufficient to determine a non-pathological beam, provided that these spectra are *sympathetic*, i.e. provided that they are such as to yield two other spectra from the quadratic identity.

Having established the interlacing of $\{\mu_n\}_1^{N-1}$ and $\{\lambda_n^{(N)}\}_1^{N-2}$, i.e.

$$\mu_1 < \lambda_1^{(N)} < \ldots < \lambda_{N-2}^{(N)} < \mu_{N-1}, \tag{3.53}$$

we can establish similarly that

$$\omega_1 < \sigma_1^{(N)} < \ldots < \omega_{N-1} < \sigma_{N-1}^{(N)} \tag{3.54}$$

and

$$\sigma_1^{(N)} < \lambda_1^{(N)} < \ldots < \lambda_{N-2}^{(N)} < \sigma_{N-1}^{(N)}. \tag{3.55}$$

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To these interlacing relations, we can add those that the data must satisfy, i.e. the small interlacing (3.41), or rather

$$\omega_1 < \nu_1 < \mu_1 < \ldots < \omega_{N-1} < \nu_{N-1} < \mu_{N-1}. \tag{3.56}$$

However, the grand interlacing (3.40) is equivalent to nine interlacing conditions; we are missing the following three:

$$\sigma_1^{(N)} < \nu_1 < \ldots < \sigma_{N-1}^{(N)} < \nu_{N-1}, \tag{3.57a}$$

$$\sigma_1^{(N)} < \mu_1 < \ldots < \sigma_{N-1}^{(N)} < \mu_{N-1}, \tag{3.57b}$$

and

$$\nu_1 < \lambda_1^{(N)} < \ldots < \lambda_{N-2}^{(N)} < \nu_{N-1}. \tag{3.57c}$$

These three interlacing conditions are related to each other. In fact, (3.57*c*) implies (3.57*a*) and (3.57*b*). One can see this by writing the quadratic identity as follows:

$$I_0(\nu_n^2)\,D_0(\nu_n^2) = Y_0(\nu_n^2)\,K_0(\nu_n^2). \tag{3.58}$$

Then (3.57*c*), together with (3.56), implies that $K_0(\nu_n^2)$ alternates in sign, i.e. (3.57*a*) holds. In addition, (3.57*a*) and (3.54) imply (3.57*b*). Thus, we only need to focus our attention on (3.57*c*). This interlacing condition *cannot* be deduced from (3.56). Therefore, *not all interlacing sequences* $\{\omega_n\}_1^{N-1}$, $\{\nu_n\}_1^{N-1}$ and $\{\mu_n\}_1^{N-1}$ *are bona fide spectra of an N-beam.* (This is to be contrasted with the situation for an N-string.) To be such, these sequences must satisfy the additional constraints

$$(-1)^n I_0(\nu_n^2) < 0, \quad n = 1, 2, \ldots, N-1, \tag{3.59}$$

or, in terms of the data,

$$(-1)^n \prod_{j=1}^{N-1} \left[\prod_{i=1}^{N-1}\left(1-\frac{\mu_j^2}{\nu_i^2}\right)^2 \prod_{i\neq j}^{N-1}\left(1-\frac{\nu_n^2}{\mu_i^2}\right)\right] \Big/ \left[\prod_{i=1}^{N-1}\left(1-\frac{\mu_j^2}{\omega_i^2}\right)\prod_{i\neq j}^{N-1}\left(1-\frac{\mu_j^2}{\mu_i^2}\right)\right] < 0.$$

3.6. *The stripping procedure*

Let us assume that we have determined $I_0(\omega^2)$ and $K_0(\omega^2)$, and that $\lambda_n^{(N)}$ and $\sigma_n^{(N)}$ thus found satisfy all the necessary interlacing conditions. The next step consists of writing (3.45) for $i = 1$ as follows:

$$-\Theta_0(\omega^2)/K_0(\omega^2) = l_0 - \Theta_1(\omega^2)/K_0(\omega^2), \tag{3.60a}$$

$$Y_1(\omega^2) = Y_0(\omega^2) + l_0\,\Theta_0(\omega^2) + l_0\,\Theta_1(\omega^2), \tag{3.60b}$$

$$-I_0(\omega^2)/Y_1(\omega^2) = f_1 - I_1(\omega^2)/Y_1(\omega^2), \tag{3.60c}$$

$$D_0(\omega^2)/Y_1(\omega^2) = m_1\omega^2 + D_1(\omega^2)/Y_1(\omega^2), \tag{3.60d}$$

$$K_1(\omega^2) = K_0(\omega^2) + f_1 D_1(\omega^2) - m_1\omega^2 I_0(\omega^2). \tag{3.60e}$$

If we divide $-\Theta_0(\omega^2)$ by $K_0(\omega^2)$, these being two polynomials of the same degree, the quotient is a constant l_0 and the remainder a polynomial of degree $N-2$ in ω^2, namely $\Theta_1(\omega^2)$. Knowing l_0 and $\Theta_1(\omega^2)$, we can find $Y_1(\omega^2)$. Dividing next $I_0(\omega^2)$ by $-Y_1(\omega^2)$, these once again being two polynomials of the same degree, we infer the quotient f_1 and the remainder $I_1(\omega^2)$. Similarly, by dividing $D_0(\omega^2)$ by $Y_1(\omega^2)$ which is of lower degree, we obtain the quotient $m_1\omega^2$ and the remainder $D_1(\omega^2)$. Finally, we can evaluate $K_1(\omega^2)$ and start the cycle over again.

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We postpone a discussion of this formal construction procedure until later and simply remark at this stage that by setting $\omega = 0$ in (3.60*a*), (3.60*b*), (3.60*d*) and (3.60*e*) we would get

$$\left.\begin{aligned} D_1(0) &= 1,\\ K_1(0) &= -F_0^{(N)}+f_1 = -\sum_2^N f_i,\\ \Theta_1(0) &= F_1-l_0F_0^{(N)} = \sum_2^N (x_i-l_0)f_i,\\ Y_1(0) &= -F_2+2l_0F_1-l_0^2F_0^{(N)},\\ &= -\sum_2^N (x_i-l_0)^2 f_i, \end{aligned}\right\} \tag{3.61}$$

where
$$F_0^{(N)} = \sum_1^N f_i.$$

The function $I_1(0)$ can be deduced from the quadratic identity

$$I_1(0) = \sum_2^N f_i \sum_2^N (x_i-l_0)^2 f_i - \left[\sum_2^N (x_i-l_0)\,f_i\right]^2.$$

These expressions are similar to those for $D_0(0), \ldots, I_0(0)$ except for the fact that the first segment as well as the first clothes-pin are missing. Thus, by means of this procedure we have 'stripped-off' a small segment of the beam. This is reminiscent of other inverse problems such as in seismic prospecting where this stripping-off is usually made in the time domain (Berkhout & van Wulfften Palthe 1979).

3.7. *The Stieltjes theorem*

The procedure we have outlined is formal since it does not necessarily yield values of l_i, m_i and f_i that are positive and hence physically meaningful. Indeed, numerical experiments made with spectra satisfying the grand interlacing have revealed that conditions (3.41) and (3.59) are *not* sufficient to guarantee the positivity of the physical characteristics of the N-beam. In contrast, the interlacing (3.2) is sufficient to guarantee the positivity of the density of the corresponding vibrating string. This is a direct result of a theorem of Stieltjes which we can state as follows.

Part 1. If $0 < \beta_1 < \alpha_1 < \ldots < \beta_{N-1} < \alpha_{N-1}$, then

$$\prod_{n=1}^{N-1}\left(1-\frac{\omega^2}{\alpha_n^2}\right)\Big/\prod_{n=1}^{N-1}\left(1-\frac{\omega^2}{\beta_n^2}\right) = l+\left[\prod_{n=1}^{N-2}\left(1-\frac{\omega^2}{\gamma_n^2}\right)\Big/\prod_{n=1}^{N-1}\left(1-\frac{\omega^2}{\beta_n^2}\right)\right], \tag{3.62}$$

where
$$l > 0$$
and
$$\alpha_i < \gamma_i < \beta_{i+1}, \quad i = 1, 2, \ldots, N-2. \tag{3.63}$$

Part 2. If $0 < \beta_1 < \gamma_1 < \ldots < \gamma_{N-2} < \beta_{N-1}$, then

$$\prod_{n=1}^{N-1}\left(1-\frac{\omega^2}{\beta_n^2}\right)\Big/\prod_{n=1}^{N-2}\left(1-\frac{\omega^2}{\gamma_n^2}\right) = -m\omega^2+\left[\prod_{n=1}^{N-2}\left(1-\frac{\omega^2}{\delta_n^2}\right)\Big/\prod_{n=1}^{N-2}\left(1-\frac{\omega^2}{\gamma_n^2}\right)\right], \tag{3.64}$$

where
$$m > 0$$
and
$$\beta_i < \delta_i < \gamma_i, \quad i = 1, 2, \ldots, N-2. \tag{3.65}$$

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We can apply part 1 of Stieltjes's theorem to (3.60*a*) to show that $l_0 > 0$. In fact

$$l_0 = \frac{F_1}{F_0^{(N)}} \prod_{n=1}^{N-1} \frac{\sigma_n^{(N)2}}{\nu_n^2} \tag{3.66}$$

or, equivalently,

$$l_0 = \frac{F_2}{F_1} \prod_{n=1}^{N-1} \frac{\nu_n^2}{\mu_n^2}. \tag{3.67}$$

Both of the above expressions for l_0 are compatible on account of the quadratic identity. Also, if we denote the zeros of $\Theta_1(\omega^2)$ by $\{\nu_n'^2\}_1^{N-2}$, then (3.63) implies that

$$\nu_i < \nu_i' < \sigma_{i+1}, \quad i = 1, 2, \ldots, N-2. \tag{3.68}$$

Eliminating l_0 from (3.60*b*), we get

$$Y_1(\omega^2) = Y_0(\omega^2) - \frac{\Theta_0^2(\omega^2)}{K_0(\omega^2)} + \frac{\Theta_1^2(\omega^2)}{K_0(\omega^2)},$$

which, because of the quadratic identity, we can also write as

$$Y_1(\omega^2) = \frac{I_0(\omega^2)\, D_0(\omega^2)}{K_0(\omega^2)} + \frac{\Theta_1^2(\omega^2)}{K_0(\omega^2)}.$$

By setting $\omega = \lambda_i$ in the above equation, we deduce that

$$Y_1(\lambda_i^2) = \frac{\Theta_1^2(\lambda_i^2)}{K_0(\lambda_i^2)}, \quad i = 1, 2, \ldots, N-1.$$

Since σ_i and λ_i interlace, $K_0(\lambda_i^2)$ changes sign as i goes from 1 to $N-1$. Consequently the zeros μ_i' of $Y_1(\omega^2)$ and λ_i interlace. In fact, we can show that

$$\mu_1' < \lambda_1 < \ldots < \mu_{N-2}' < \lambda_{N-2}. \tag{3.69}$$

This is the condition that is necessary if we were to apply part 1 of Stieltjes's theorem to (3.60*c*). As a consequence it follows that $f_1 > 0$ and that

$$\lambda_i < \lambda_i' < \mu_{i+1}', \quad i = 1, 2, \ldots, N-3. \tag{3.70}$$

In fact

$$f_1 = \frac{F_0^{(N)} F_2 - F_1^2}{F_2'} \prod_{n=1}^{N-2} \frac{\mu_n'^2}{\lambda_n^2}, \tag{3.71}$$

where

$$F_2' = \sum_2^N (x_i - l_0)^2 f_i.$$

A more useful expression for f_1 can be obtained by writing (3.45*e*) thus:

$$K_1 = K_0 + f_1 D_0 - \omega^2 m_1 I_1,$$

from which we deduce that

$$f_1 = F_0^{(N)} \prod_{n=1}^{N-1} \frac{\omega_n^2}{\sigma_n^{(N)2}}. \tag{3.72}$$

Incidentally, as a result of the interlacing between ω_n and $\sigma_n^{(N)}$, f_1 is smaller than $F_0^{(N)}$. Hence $K_1(0)$ as given by (3.61) is negative. In the same vein, (3.66) shows that $l_0 F_0^{(N)}$ is smaller than F_1 and hence $\Theta_1(0)$ as given by (3.61) is positive. Finally, in this same formula, $Y_1(0)$ can be seen to be negative. In summary

$$K_1(0) < 0, \quad \Theta_1(0) > 0, \quad Y_1(0) < 0, \tag{3.73}$$

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i.e. they have the same signs as $K_0(0)$, $\Theta_0(0)$ and $Y_0(0)$ respectively. We next turn to the equation (3.60*c*) for K_1 and proceed to eliminate f_1 and m_1 from it. In other words, we write it thus:

$$K_1 = K_0 + \frac{I_1 - I_0}{Y_1} D_1 - \frac{D_0 - D_1}{Y_1} I_0,$$

which after simplifications brought about by the quadratic identity for $i = 1$, becomes

$$-D_0 I_0 + Y_1 K_0 - \Theta_1^2 = 0. \tag{3.74}$$

Setting $\omega = \mu_i'$ in this hybrid quadratic identity, we see that

$$D_0(\mu_i'^2) = -\Theta_1^2(\mu_i'^2)/I_0(\mu_i'^2).$$

Making use of (3.69) we deduce that

$$\omega_i' < \mu_i' < \omega_{i+1}, \quad i = 1, 2, \ldots, N-2. \tag{3.75}$$

We are now in a position to apply part 2 of Stieltjes's theorem to (3.60*d*). As a result $m_1 > 0$ and

$$\omega_i < \omega_i' < \mu_1', \quad i = 1, 2, \ldots, N-2. \tag{3.76}$$

In fact

$$m_1 = \frac{1}{F_2'} \left[\prod_{n=1}^{N-2} \mu_n'^2 \Big/ \prod_{n=1}^{N-1} \omega_n^2 \right]. \tag{3.77}$$

For the record, we also write an alternative formula for m_1, namely

$$m_1 = \frac{F_0^{(N)}}{F_0^{(N)} F_2 - F_1^2} \left[\prod_{n=1}^{N-2} \lambda_n^{(N)2} \Big/ \prod_{n=1}^{N-1} \sigma_n^{(N)2} \right]. \tag{3.78}$$

We can derive two more interlacings by setting ω equal to ω_i' and λ_i' in the quadratic identity. These are

$$\omega_i' < \sigma_i' < \omega_{i+1}', \quad i = 1, 2, \ldots, N-2, \tag{3.79}$$

and

$$\sigma_i' < \lambda_i' < \sigma_{i+1}', \quad i = 1, 2, \ldots, N-2. \tag{3.80}$$

In summary, starting with the polynomials $I_0(\omega^2), \ldots, D_0(\omega^2)$ whose zeros satisfy the grand interlacing (3.40) we can deduce

(i) l_0, f_1 and m_1, which are positive,

(ii) new polynomials $I_1(\omega^2), \ldots, D_1(\omega^2)$ of lower degree; these polynomials have the same structure as the original ones at $\omega = 0$. However, their zeros satisfy only *four* interlacing conditions. These conditions are

$$\left.\begin{aligned} \mu_1' &< \lambda_1' < \ldots < \lambda_{N-3}' < \mu_{N-2}', \\ \sigma_1' &< \lambda_1' < \ldots < \lambda_{N-3}' < \sigma_{N-2}', \\ \omega_1' &< \mu_1' < \ldots < \omega_{N-2}' < \mu_{N-2}', \\ \omega_1' &< \sigma_1' < \ldots < \omega_{N-2}' < \sigma_{N-2}'. \end{aligned}\right\} \tag{3.81}$$

Thus, we do not close the cycle: other conditions must be placed on the data to insure that the eigenvalues of the 'stripped' beam satisfy the grand interlacing.

3.8. *The limit $N \to \infty$*

We would like to examine whether or not the sequences $\{\rho^{(N)}(x)\}$ and $\{r^{(N)}(x)\}$ converge, where

$$\rho^{(N)}(x) = \sum_{i=1}^{N-1} m_i^{(N)} \delta(x - x^{(N)}),$$

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and
$$\frac{1}{r^{(N)}(x)} = \sum_{i=1}^{N} f_i^{(N)}\,\delta(x-x^{(N)}).$$

We shall assume that the beam has a finite length, i.e.

$$L = \lim_{N\to\infty} x_N^{(N)}, \tag{3.82}$$

or, equivalently,
$$L = \lim_{N\to\infty} \sum_{i=0}^{N-1} l_i^{(N)}, \tag{3.82'}$$

where
$$l_0^{(N)} = \frac{F_2}{F_1} \prod_{n=1}^{N-1} \frac{\nu_n^2}{\mu_n^2},$$
$$l_1^{(N)} = \frac{F_2'}{F_1'} \prod_{n=1}^{N-2} \frac{\nu_n'^2}{\mu_n'^2} \quad \text{etc.}$$

If the data are such that (3.82) holds, then we can define

$$\mathscr{F}(x) = \int_x^L \frac{\mathrm{d}t}{r(t)} \tag{3.83}$$

and
$$\mathscr{M}(x) = \int_x^L (L-t)^2 \rho(t)\,\mathrm{d}t. \tag{3.84}$$

Clearly, the Nth approximations of these functions are

$$\mathscr{F}^{(N)}(x) = \sum_{x_i\geqslant x}^{N} f_i^{(N)} \tag{3.83'}$$

and
$$\mathscr{M}^{(N)}(x) = \sum_{x_i\geqslant x}^{N} (x_N^{(N)}-x_i^{(N)})^2 m_i^{(N)}. \tag{3.84'}$$

From their definitions, $\mathscr{F}^{(N)}(x)$ and $\mathscr{M}^{(N)}(x)$ are non-increasing functions of x. In fact

$$\mathscr{F}^{(N)}(x) \leqslant F_0^{(N)}. \tag{3.85}$$

We can express $F_0^{(N)}$ in terms of the data by writing (3.52) as follows:

$$F_0^{(N)} = \frac{F_1^2}{F_2} + \frac{I_0(0)}{F_2},$$

or
$$F_0^{(N)} = \frac{F_1^2}{F_2} + \frac{F_1^2}{F_2} \sum_{n=1}^{N-1} \left[\prod_{k=1}^{N-1} \left|1-\frac{\mu_n^2}{\nu_k^2}\right|^2 \Big/ \prod_{k=1}^{N-1} \left|1-\frac{\mu_n^2}{\omega_k^2}\right| \prod_{k\neq n}^{N-1} \left|1-\frac{\mu_n^2}{\mu_k^2}\right| \right]. \tag{3.86}$$

Therefore, if
$$\sum_{n=1}^{\infty} \left[\prod_{k=1}^{\infty} \left|1-\frac{\mu_n^2}{\nu_k^2}\right|^2 \Big/ \prod_{k=1}^{\infty} \left|1-\frac{\mu_n^2}{\omega_k^2}\right| \prod_{k\neq n}^{\infty} \left|1-\frac{\mu_n^2}{\mu_k^2}\right| \right] < \infty, \tag{3.87}$$

then the limit of $F_0^{(N)}$ as N tends to infinity exists, i.e.

$$F_0 = \lim_{N\to\infty} F_0^{(N)} \tag{3.88}$$

and this limit provides an upper bound for $\mathscr{F}^{(N)}(x)$, i.e.

$$\mathscr{F}^{(N)}(x) \leqslant F_0 \quad \text{for} \quad 0 \leqslant x \leqslant L \quad \text{and for every } N. \tag{3.89}$$

Similarly, the functions $\{\mathscr{M}^{(N)}(x)\}$ are also bounded above. We can see this as follows. Let us substitute the series representations for $Y(x, \omega^2)$ and $D(x, \omega^2)$, namely

$$Y(x, \omega^2) = -r^2(L) \left[\int_x^L \frac{(t-x)^2}{r}\,\mathrm{d}t + \sum_{n=1}^{\infty} (-1)^n y_n(x)\,\omega^{2n} \right]$$

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and $$D(x, \omega^2) = r^2(L)\left[1+\sum_{n=1}^{\infty}(-1)^n d_n(x)\,\omega^{2n}\right]$$

in the auxiliary equation (3.25f). Then the linear terms in ω^2 yield

$$d_1' = -\rho\int_x^L \frac{(t-x)^2}{r(t)}\,\mathrm{d}t,$$

i.e. $$d_1(x) = \int_x^L \rho(t')\,\mathrm{d}t'\int_x^L \frac{(t-t')^2}{r(t)}\,\mathrm{d}t. \tag{3.90}$$

Confronting this result with the product representation of $D(x, \omega^2)$ given in (3.27′e), we conclude that

$$\sum_{n=1}^{\infty}\frac{1}{\omega_n^2} = \int_0^L \rho(t')\,\mathrm{d}t'\int_{t'}^L \frac{(t-t')^2}{r(t)}\,\mathrm{d}t$$

or, after interchanging the order of integration,

$$\sum_{n=1}^{\infty}\frac{1}{\omega_n^2} = \int_0^L \frac{\mathrm{d}t}{r(t)}\int_0^t (t-x)^2\rho(x)\,\mathrm{d}x. \tag{3.91}$$

Since $1/r(t)$ is integrable for t in $(0, L)$ we conclude that

$$\int_0^t (t-x)^2\rho(x)\,\mathrm{d}x < \infty.$$

As a result $\mathscr{M}(0)$ exists and

$$\mathscr{M}^{(N)}(x) \leqslant \mathscr{M}(0) \quad \text{for} \quad 0 \leqslant x \leqslant L \quad \text{and for all } N. \tag{3.92}$$

Therefore, the non-increasing sequences of functions $\{\mathscr{F}^{(N)}(x)\}$ and $\{\mathscr{M}^{(N)}(x)\}$ are bounded above. Hence, according to Helly's selection theorem (see, for example, Natanson 1955, p. 220), we can extract from these sequences two subsequences that converge for all x in $(0, L)$ to two non-increasing functions. In view of the uniqueness theorem, these limit functions must be the functions $\mathscr{F}(x)$ and $\mathscr{M}(x)$ associated with the non-pathological beam. Finally, $\rho(x)$ and $r(x)$ are obtained after differentiation.

4. Concluding remarks

We saw that the information contained in the impulse response is equivalent to the three sympathetic spectra $\{\omega_n\}$, $\{\nu_n\}$ and $\{\mu_n\}$, and the two moments of the flaccidity F_1 and F_2. We also saw that the information contained in the impulse response does not, strictly, determine a single beam. Rather, it determines a class of beams: all the beams in this class have the same oscillating portion, but they differ over that rear portion of their lengths which is stationary. This stationary portion is made up of a weightless rod of infinite rigidity. I was surprised by the fact that the massive wall in which the beams are embedded is not part of the ambiguity.

The uniqueness proof has the drawback of requiring that $\rho(x)$ and $r(x)$ be differentiable functions. This is doubly regrettable: first because the degree of smoothness of $\rho(x)$ and $r(x)$ is not easy to infer from the data and secondly because discontinuities are likely to occur in the more difficult geophysical problem for which the beam is to serve as a guide. Ideally, these smoothness requirements ought to be relaxed. To that effect, it would be desirable to derive the uniqueness result from the construction, i.e. to show that the approximations form a Cauchy sequence.

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Our discussion of the inverse problem was made entirely in the frequency domain. Yet the stripping procedure, which unravelled progressively the structure of the beam, is a reminder of the presence of a wave propagating along the beam. In spite of this time-domain intrusion, I believe that the frequency approach is best suited for the inverse problem at hand. Indeed, the continued fractions reliance on the ω-dependence shows very clearly the advantage of working in the frequency domain. In addition, the theories of entire functions and of oscillating kernels provide some very powerful tools for the investigation of the problem in the frequency domain.

The actual construction of the solution required a discretization of the original problem. This discretization is an essential step which cannot be by-passed. Indeed, the global condition (3.87) is a direct result of this apparent detour. The same remark holds for the vibrating string: the global condition (3.3) is a consequence of a discretization followed by the limiting process $N \to \infty$.

The existence of the solution to the inverse problem is tied to very stringent interlacing conditions on $\{\omega_n\}$, $\{\nu_n\}$ and $\{\mu_n\}$. These conditions are such as to guarantee that the spectra $\{\omega_n(x)\}, \ldots, \{\lambda_n(x)\}$ for the truncated beam spanning the interval (x, L) satisfy the grand interlacing. Rather than having to check at each step of the construction whether the grand interlacing is satisfied, it would be desirable to have explicit ways of testing, from the outset, whether the data are bona fide.

In the investigation of the limiting process $N \to \infty$, the length L of the beam was assumed to be finite. This convenient assumption is not essential. Very minor modifications are needed to handle the infinite beam since the Helly selection theorem can still be used.

Finally, our analysis dealt exclusively with the beam in the free–clamped configuration. For other vibrating configurations, the details of the results would be modified: for instance, the impulse responses would be equivalent to other trios of sympathetic spectra, and the gross constants would not necessarily be related to moments of the flaccidity. However, the approach to the questions of uniqueness, existence and construction that was used in the present paper ought to be applicable for general boundary conditions.

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